

Matrix semigroups with constant spectral radius ^{*}

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Abstract

Multiplicative matrix semigroups with constant spectral radius (c.s.r.) are studied and applied to several problems of algebra, combinatorics, functional equations, and dynamical systems. We show that all such semigroups are characterized by means of irreducible ones. Each irreducible c.s.r. semigroup defines walks on Euclidean sphere, all its nonsingular elements are similar (in the same basis) to orthogonal. We classify all nonnegative c.s.r. semigroups and arbitrary low-dimensional semigroups. For higher dimensions, we describe five classes and leave an open problem on completeness of that list. The problem of algorithmic recognition of c.s.r. property is proved to be polynomially solvable for irreducible semigroups and undecidable for reducible ones.

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1. Introduction

We consider multiplicative closed semigroups of real $d \times d$ matrices. For a nonempty compact family \mathcal{A} of $d \times d$ matrices, we denote by $\mathcal{S}_{\mathcal{A}}$ the semigroup generated by \mathcal{A} by multiplications and taking closure.

Definition 1 *A matrix semigroup \mathcal{S} has constant spectral radius (in short, \mathcal{S} is c.s.r.) if the spectral radius of all its elements is the same and nonzero. A compact matrix family \mathcal{A} is called c.s.r. if it generates a c.s.r. semigroup $\mathcal{S}_{\mathcal{A}}$.*

The spectral radius $\rho(A)$ of a matrix A is the maximal modulus of its eigenvalues. Clearly, in any c.s.r. semigroup the spectral radius of all matrices equals 1.

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Definition 2 A family of matrices \mathcal{A} is irreducible if there is no proper linear subspace of \mathbb{R}^d invariant for all matrices from \mathcal{A} .

The semigroup $\mathcal{S}_{\mathcal{A}}$ generated by a family \mathcal{A} is irreducible precisely when so is \mathcal{A} . Semigroups with constant spectral radius have several important characteristic properties listed below. Some of them concern only irreducible semigroups.

1. If \mathcal{S} is a semigroup with multiplicative spectral radius, i.e., $\rho(AB) = \rho(A)\rho(B)$, $A, B \in \mathcal{S}$, then the semigroup $\{[\rho(A)]^{-1}A \mid A \in \mathcal{S}, \rho(A) \neq 0\}$ has constant spectral radius. Thus, the study of semigroups with multiplicative spectral radius is essentially reduced to c.s.r. semigroups. This observation was put to good use in [24]. Moreover, for irreducible semigroups, the submultiplicativity of the spectral radius ($\rho(AB) \leq \rho(A)\rho(B)$, $A, B \in \mathcal{S}$) is equivalent to its multiplicativity [16, Theorem 2.1]. This property of semigroups can be relaxed further to the so-called Rota condition [24, Theorem 4.3].

2. All finite matrix semigroups that do not contain zero matrices are c.s.r. For integer matrices, the converse is also true: every irreducible c.s.r. semigroup of integer matrices is finite. We analyse this aspect in more detail in Subsection 9.1.

3. For every irreducible c.s.r. semigroup \mathcal{S} , there exists a norm in \mathbb{R}^d such that the induced operator norm of all matrices from \mathcal{S} is one [24, Theorem 4.7]. Since all matrix norms are equivalent, we obtain the following criterion:

An irreducible semigroup \mathcal{S} is c.s.r. if and only if there are two positive constants C_1, C_2 such that $C_1 \leq \|A\| \leq C_2$ for all $A \in \mathcal{S}$.

Thus, for irreducible semigroups, the c.s.r. property is equivalent to boundedness from above and from below.

4. Let \mathcal{A} be a compact family of matrices. The boundedness property (item 3) of the semigroup $\mathcal{S}_{\mathcal{A}}$ means equal asymptotic behavior of products $\Pi = A_n \cdots A_1$ for all possible $A_j \in \mathcal{A}$, $j = 1, \dots, n$, $n \in \mathbb{N}$. Thus, *if \mathcal{A} is irreducible, then the semigroup $\mathcal{S}_{\mathcal{A}}$ is c.s.r. if and only if $C_1 \leq \|\Pi\| \leq C_2$ for all products Π of matrices from \mathcal{A} .*

For finite families \mathcal{A} , this property is better expressed in terms of their p -radii:

Definition 3 For a finite family $\mathcal{A} = \{A_1, \dots, A_m\}$ and for a given $p \in \mathbb{R} \cup \{\pm\infty\}$ the p -radius $\rho_p(\mathcal{A})$ is defined as follows:

$$\rho_p = \lim_{k \rightarrow \infty} \left(m^{-p} \sum_{d_1, \dots, d_k} \|A_{d_k} \cdots A_{d_1}\|^p \right)^{\frac{1}{pk}} \quad (1)$$

with the usual modifications in the cases $p = 0$ and $p = \pm\infty$.

If there is a zero product of matrices from \mathcal{A} , then for $p \leq 0$, we set $\rho_p = 0$. The p -radius is a non-decreasing function in p . The p -radii are also called *joint spectral characteristics* of \mathcal{A} . For one matrix, i.e., in case $m = 1$, $\mathcal{A} = \{A\}$, they all coincide with the spectral radius of A . For a family of matrices, they are always different, apart from the case when the family $r^{-1}\mathcal{A} = \{r^{-1}A_1, \dots, r^{-1}A_m\}$ is c.s.r. for some number r :

The p -radii of a family \mathcal{A} coincide for all p if and only if the family $r^{-1}\mathcal{A}$ is c.s.r. for some $r > 0$. In this case $\rho_p(\mathcal{A}) = r$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$.

This is proved in the next section (Proposition 1), along with more details of the joint spectral characteristics.

Thus, the c.s.r. property of a family \mathcal{A} means equal asymptotic behaviour of all products of matrices from \mathcal{A} . Such families and semigroups play a special role in many applications, where nonhomogeneous matrix products are used (see Section 9). The simplest example of the c.s.r. property is any semigroup of orthogonal matrices. Another example is a semigroup of stochastic matrices (such semigroups are, however, reducible). Other nontrivial examples were analyzed in [16, 22, 24, 25] along with many properties of c.s.r. semigroup.

In this paper, we begin with characterizing arbitrary c.s.r. semigroups by means of irreducible semigroups (Theorem 1). Then we analyze irreducible c.s.r. semigroups. Theorem 2 shows that all nonsingular elements of an irreducible c.s.r. semigroup are orthogonal matrices in some (common) basis. In particular, all c.s.r. semigroups of nonsingular matrices are, up to a linear transform, subgroups of $O(d)$. So, this case is simple. However, the problem of classifying all c.s.r. semigroups (possibly containing singular matrices) is surprisingly nontrivial. We prove that an irreducible semigroup \mathcal{S} is c.s.r. if and only if it defines walks on an ellipsoid, i.e., there exists a point $x \in \mathbb{R}^d$, $x \neq 0$, such that its orbit $\{Ax \mid A \in \mathcal{S}\}$ lies on the surface of an ellipsoid (Theorem 3). This theorem asserts only the existence of such a point x and does not give any recipe how to find it, along with the ellipsoid, for a given semigroup \mathcal{S} . Note also that the ellipsoid in this theorem cannot be replaced by another convex body. In the special case, when all matrices are nonnegative, a complete classification of c.s.r. semigroups is obtained in Section 6. For the general case, we describe five classes of c.s.r. semigroups and leave an open problem that this classification is complete. The affirmative answer is proved for low dimensions, in Section 8. Since for higher dimensions, the problem is unsolved, it is natural to address the question of algorithmical recognition of c.s.r. semigroups. Given a finite set \mathcal{A} of rational matrices, one needs to decide whether the semigroup $\mathcal{S}_{\mathcal{A}}$ is c.s.r. The answer is rather curious: for irreducible \mathcal{A} , there is a polynomial time algorithm (we present it in Section 7), while for reducible \mathcal{A} this problem is algorithmically undecidable, even if \mathcal{A} is a pair of nonnegative matrices.

Finally, in Section 9, we consider applications. Our results are applied to five different problems. In §9.1 we analyze finite matrix semigroups. We prove that there is a polynomial time algorithm that for a finite set of integer matrices, under some mild assumption, decides whether it generates a finite semigroup. In §9.2 we consider linear switching systems of ODE and describe systems with equal asymptotic growth of trajectories for all switching laws. Then we apply c.s.r. semigroups to the study of fractal curves, including refinable functions and wavelets. The main result describes fractal curves with constant local regularity. Finally, in §9.5, we solve an open problem formulated by B.Reznick (1990) on the asymptotics of the Euler binary partition function.

Throughout the paper, we denote by (\cdot, \cdot) the standard inner product in \mathbb{R}^d , by $\|x\|_2 = \sqrt{(x, x)}$ the Euclidean norm, by $O(d)$ the orthogonal group, by \mathbf{S}^{d-1} the unit Euclidean sphere in \mathbb{R}^d . We assume a basis in \mathbb{R}^d is fixed and identify a linear operator with the corresponding matrix.

2. The c.s.r. property and the joint spectral characteristics

The joint spectral characteristics of finite matrix sets such as Lyapunov exponents, joint and lower spectral radii, etc., have a rich history and numerous applications (see [4, 7, 10, 11] and references therein). In view of Definition 3, all joint spectral characteristics form a one-parametric family (the p -radii). Each of them indicates the asymptotic growth of the L_p -mean of norms for products of matrices of length k as $k \rightarrow \infty$. The limit (1) always exists and does not depend on the matrix norm [26]. For $p = 0$, formula (1) is modified as $\rho_0 = \lim_{k \rightarrow \infty} \left(\prod_{d_1, \dots, d_k} \|A_{d_k} \cdots A_{d_1}\| \right)^{\frac{1}{k2^k}}$, this is the *Lyapunov exponent*. For $p = \pm\infty$ we obtain the *joint spectral radius* and the *lower spectral radius* respectively:

$$\rho_{+\infty} = \lim_{k \rightarrow \infty} \max_{d_1, \dots, d_k} \|A_{d_k} \cdots A_{d_1}\|^{1/k}; \quad \rho_{-\infty} = \lim_{k \rightarrow \infty} \min_{d_1, \dots, d_k} \|A_{d_k} \cdots A_{d_1}\|^{1/k}. \quad (2)$$

The p -radius is a non-decreasing (typically, strictly increasing) function in p . However, there are exceptions, when all p -radii coincide. For example, in case of one matrix $\mathcal{A} = \{A\}$, we have $\rho_p = \rho(A)$ for all p . The following result describes all families with equal p -radii:

Proposition 1 *We have $\rho_p(\mathcal{A}) = r > 0$ for all $p \in \mathbb{R} \cup \{\pm\infty\}$, if and only if the semigroup generated by the family $r^{-1}\mathcal{A} = \{r^{-1}A_1, \dots, r^{-1}A_m\}$ is c.s.r.*

Proof. It suffices to consider the case $r = 1$. The following assertions are well-known:

$$\rho_{-\infty} \leq \min_{d_1, \dots, d_k} \rho^{1/k}(A_{d_k} \cdots A_{d_1}) \leq \max_{d_1, \dots, d_k} \rho^{1/k}(A_{d_k} \cdots A_{d_1}) \leq \rho_{+\infty}, \quad k \in \mathbb{N}, \quad (3)$$

and

$$\min_{d_1, \dots, d_k} \rho^{1/k}(A_{d_k} \cdots A_{d_1}) \rightarrow \rho_{-\infty}; \quad \max_{d_1, \dots, d_k} \rho^{1/k}(A_{d_k} \cdots A_{d_1}) \rightarrow \rho_{+\infty} \quad \text{as } k \rightarrow \infty \quad (4)$$

(see, for instance, [10]). If $\rho_{-\infty} = \rho_{+\infty} = 1$, then $\rho(A_{d_k} \cdots A_{d_1}) = 1$ for every product of matrices from \mathcal{A} , hence $\mathcal{S}_{\mathcal{A}}$ is c.s.r. Conversely, if $\mathcal{S}_{\mathcal{A}}$ is c.s.r., then $\rho(A_{d_k} \cdots A_{d_1}) = 1$ for each product. Substituting in (4) and taking limit as $k \rightarrow \infty$, we obtain $\rho_{-\infty} = \rho_{+\infty} = 1$. \square

The notions of the joint and lower spectral radii are directly extended to arbitrary compact families \mathcal{A} , with the same properties including (3) and (4). In the sequel we use the short standard notation for them: $\rho_{+\infty}(\mathcal{A}) = \rho(\mathcal{A})$ (the joint spectral radius) and $\rho_{-\infty}(\mathcal{A}) = \check{\rho}(\mathcal{A})$ (the lower spectral radius). Note that they are well-defined only for compact families \mathcal{A} .

3. Arbitrary c.s.r. semigroups vs irreducible semigroups

Most of known results on semigroups with constant and with multiplicative spectral radius deal with irreducible semigroups [16, 22, 24, 25]. In this section we show that the

general case can be characterized by the irreducible one. The main result, Theorem 1, asserts that any reducible c.s.r. semigroup can be factored to several irreducible semigroups of smaller dimensions, one of which is c.s.r. and the others are contractions.

Let us begin with several auxiliary results. We say that a matrix semigroup \mathcal{S} has a *bounded spectrum* if there is a constant C such that $\rho(A) \leq C$ for all $A \in \mathcal{A}$.

Proposition 2 *An irreducible matrix semigroup has a bounded spectrum if and only if it is bounded.*

The proof is in Appendix. This result generalizes [24, Theorem 4.1], where it was proved that any irreducible c.s.r. semigroup is bounded. Note that the irreducibility assumption is essential, the corresponding example is the semigroup generated by one matrix A such that $(A)_{21} = 0$ and all other entries are ones. An important consequence is that the joint spectral radius $\rho(\mathcal{S})$ is well-defined for any irreducible closed semigroup \mathcal{S} with bounded spectrum. Moreover, $\rho(\mathcal{S}) \leq 1$, since all products of matrices from \mathcal{S} are bounded.

Proposition 3 *For any irreducible matrix semigroup \mathcal{S} with a bounded spectrum, there is a norm $\|\cdot\|$ in \mathbb{R}^d such that, in the induced operator norm, we have $\|A\| \leq 1$, $A \in \mathcal{S}$.*

Proof. By Proposition 2, the semigroup \mathcal{S} is bounded, hence $\rho(\mathcal{S})$ is well-defined and does not exceed one. Since \mathcal{S} is irreducible, there exists Barabanov's norm for \mathcal{S} in \mathbb{R}^d , for which $\|A\| \leq \rho(\mathcal{S})$, $A \in \mathcal{S}$ (see [1]). This completes the proof. \square

For any reducible semigroup \mathcal{S} , there exists a basis in \mathbb{R}^d in which every matrix $A \in \mathcal{S}$ has the following block upper triangular form:

$$A = \begin{pmatrix} A^{(1)} & * & \dots & * \\ 0 & A^{(2)} & * & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & A^{(s)} \end{pmatrix} \quad (5)$$

with square diagonal blocks of sizes $d^{(i)}$, $i = 1, \dots, s$, $\sum_{i=1}^s d^{(i)} = d$. The locations of diagonal blocks and their sizes are the same for all matrices $A \in \mathcal{S}$. For each $i = 1, \dots, s$, the semigroup $\mathcal{S}^{(i)}$ formed by $d^{(i)} \times d^{(i)}$ matrices of the i th block is irreducible. If \mathcal{S} is irreducible, we set $s = 1$, $d^{(1)} = d$, $\mathcal{S}^{(1)} = \mathcal{S}$.

For every family of matrices \mathcal{A} , not necessarily a semigroup, we use the same factorization (5) with irreducible families $\mathcal{A}^{(i)}$ of matrices in the i th diagonal blocks. Clearly, $\rho(A) = \max_{i=1, \dots, s} \rho(A_i)$. If all families $\mathcal{A}^{(i)}$ are compact (and hence, the joint spectral radius is well-defined), then $\rho(\mathcal{A}) = \max_{i=1, \dots, s} \rho(\mathcal{A}_i)$ (see [7]).

Theorem 1 *A matrix semigroup \mathcal{S} is c.s.r. if and only if, in factorization (5), one of the irreducible semigroups $\mathcal{S}^{(j)}$ is c.s.r., and the others satisfy $\rho(\mathcal{S}^{(i)}) \leq 1$.*

Remark 1 Since all semigroups \mathcal{S}_i are irreducible and have bounded spectra (because $\rho(A^{(i)}) \leq \rho(A) = 1$, $A \in \mathcal{S}$), Proposition 2 implies that they are all compact. Therefore, $\rho(\mathcal{S}^{(i)})$ is well-defined.

For a semigroup $\mathcal{S}_{\mathcal{A}}$ generated by a matrix family \mathcal{A} , Theorem 1 is formulated in the same way: \mathcal{A} generates a c.s.r. semigroup if and only if, in factorization (5), one of the irreducible families $\mathcal{A}^{(j)}$ generates a c.s.r. semigroup, and the others satisfy $\rho(\mathcal{A}^{(i)}) \leq 1$.

Proof of Theorem 1. Sufficiency is clear: if \mathcal{S} is factored to the form (5) with a c.s.r. semigroup in the j th diagonal block, then, for every $A \in \mathcal{S}$ we have $\rho(A) = \max_{i=1,\dots,s} \rho(A_i) = \rho(A_j) = 1$, and hence \mathcal{S} is c.s.r. Indeed, $\rho(A_j) = 1$, since $\mathcal{S}^{(j)}$ is c.s.r., and $\rho(A_i) \leq \rho(\mathcal{S}^{(i)}) \leq 1$, for $i \neq j$.

Necessity. If \mathcal{S} is c.s.r., then all semigroups $\mathcal{S}^{(i)}$ have bounded spectra ($\rho(A^{(i)}) \leq \rho(A) = 1$, $A \in \mathcal{S}$), and hence, by Proposition 3, the norms of all matrices of these semigroups are bounded uniformly by some constant C . If some semigroup \mathcal{S}_j is c.s.r., then the proof is completed. If, otherwise, for every $i = 1, \dots, s$, there is a matrix $A_i \in \mathcal{S}$, whose i th block $A_i^{(i)}$ has its spectral radius smaller than 1, then, for an arbitrary n , we consider the product $\Pi = A_1^n \cdots A_s^n \in \mathcal{S}$. If n is large enough, then, $\|(A_i^{(i)})^n\| < 1/C^2$, for every $i = 1, \dots, s$. Therefore, for the i th block of Π , we have

$$\begin{aligned} \|\Pi^{(i)}\| &= \|(A_1^{(i)})^n \cdots (A_s^{(i)})^n\| = \|(A_1^{(i)})^n \cdots (A_{i-1}^{(i)})^n\| \cdot \|(A_i^{(i)})^n\| \cdot \|(A_{i+1}^{(i)})^n \cdots (A_s^{(i)})^n\| \\ &\leq C \|(A_i^{(i)})^n\| C < 1. \end{aligned}$$

Thus, $\rho(\Pi^{(i)}) \leq \|\Pi^{(i)}\| < 1$, for all i , and so $\rho(\Pi) < 1$, which contradicts to the c.s.r. property of \mathcal{S} . □

Example 1 If a semigroup \mathcal{S} consists of row-stochastic matrices, then it obviously c.s.r. In this case one can take $e = (1, \dots, 1)^T \in \mathbb{R}^d$ as the first basis vector, and the other $d - 1$ basis vectors span the subspace e^\perp . In this basis, all matrices from \mathcal{S} get the form (5) with $d^{(1)} = 1$, $d^{(2)} = d - 1$, and for every $A \in \mathcal{S}$ the one-dimensional matrix $A^{(1)}$ is 1.

4. Nonsingular elements in c.s.r. semigroups

The set of nonsingular matrices in a c.s.r. semigroups has a simple structure:

Theorem 2 *If an irreducible matrix semigroup \mathcal{S} has constant spectral radius, then in a suitable basis in \mathbb{R}^d , all nonsingular elements of \mathcal{S} are orthogonal matrices.*

Thus, if \mathcal{S} is c.s.r., then there is a basis in \mathbb{R}^d in which all nonsingular matrices of \mathcal{S} are orthogonal. Note that it is easily shown that each nonsingular matrix in a c.s.r. semigroup is similar to an orthogonal one. What is nontrivial in Theorem 2 that all those similarities are realized by the same linear transform.

Proof. By Proposition 3, there is norm $\|\cdot\|$ in \mathbb{R}^d such that $\|A\| \leq 1$, $A \in \mathcal{S}$, in the induced operator norm. For the unit ball $\mathbf{B} \subset \mathbb{R}^d$ of that norm, we have $A\mathbf{B} \subset \mathbf{B}$, $A \in \mathcal{S}$. Furthermore, it was shown in [24, Theorem 2.5] that each element of an irreducible c.s.r. semigroup is a direct sum of a nilpotent operator and an operator similar to orthogonal. Hence, every nonsingular element of \mathcal{S} is similar to orthogonal. Therefore, its determinant is one, and hence this operator preserves the volume. Consequently, for every nonsingular matrix $A \in \mathcal{S}$, we have $A\mathbf{B} = \mathbf{B}$. Let now \mathbf{E} be the John ellipsoid of the convex body \mathbf{B} , i.e., an ellipsoid of the maximal volume contained in \mathbf{B} . From the uniqueness of the John ellipsoid [37] it follows that $A\mathbf{E} = \mathbf{E}$. Taking the basis, in which \mathbf{E} is a Euclidean ball, we obtain that every nonsingular element $A \in \mathcal{S}$ is orthogonal. \square

Corollary 1 *Suppose a semigroup \mathcal{S} is generated by an irreducible set of nonsingular matrices; then \mathcal{S} is c.s.r. if and only if there is a basis in \mathbb{R}^d in which all elements of \mathcal{S} are orthogonal matrices.*

Remark 2 A semigroup is generated by nonsingular matrices may contain singular elements. They may appear after taking closure.

Proof of Corollary 1. By Theorem 2 there is a basis in which all elements of the generating set are orthogonal. Hence, all their products are also orthogonal, and all limit points of these products also are. \square

Since a c.s.r. semigroup of nonsingular matrices is actually a group [24], we obtain

Corollary 2 *If a c.s.r. semigroup consists of nonsingular matrices, it is similar (in a common basis in \mathbb{R}^d) to a subgroup of $O(d)$.*

5. Irreducible c.s.r. semigroups define walks on Euclidean sphere

The results of previous section may make a wrong impression that all c.s.r. semigroups are easily classified. In fact, there is a rich variety of classes of such semigroups, the problem of their complete classification is still unsolved (see Section 8). In view of Theorem 2, this is caused by singular matrices in c.s.r. semigroups. The following criterion is actually a weaker version of Theorem 2 that holds in general case, without the non-singularity assumption.

Theorem 3 *An irreducible semigroup \mathcal{S} is c.s.r. precisely when there is a suitable basis and a point $x \in \mathbb{R}^d$ such that $\|x\|_2 = 1$ and $\|Ax\|_2 = 1$, for all $A \in \mathcal{S}$.*

Proof. *Sufficiency.* By irreducibility, the set $\{Ax \mid A \in \mathcal{S}\}$ contains linearly independent vectors $\{x_i\}_{i=1}^d$. Since $\|Ax_i\|_2 = 1$ for every $A \in \mathcal{S}$, it follows that in the basis $\{x_i\}_{i=1}^d$ all the matrices from \mathcal{S} are uniformly bounded, hence $\rho(\mathcal{S}) \leq 1$. On the other hand, if $\rho(A) < 1$ for some $A \in \mathcal{S}$, then $\|A^n x\|_2 < 1$, for sufficiently large n , which contradicts the assumption. Thus, $\rho(A) = 1$ for all $A \in \mathcal{S}$.

Necessity. We prove under the assumption that \mathcal{S} is finitely generated; the general case then follow by the standard compactness argument. Thus, assume \mathcal{S} is generated by a set $\mathcal{A} = \{A_1, \dots, A_m\}$. Since \mathcal{S} is c.s.r. it follows that $\rho_1(\mathcal{A}) = \rho_2(\mathcal{A}) = 1$. For an irreducible set \mathcal{A} , there exists an ellipsoidal norm $\|\cdot\|$ such that $\left(\frac{1}{m} \sum_{i=1}^m \|A_i z\|_2^2\right)^{1/2} = \rho_2(\mathcal{A}) \|z\|$ for every $z \in \mathbb{R}^d$ [26]. In a suitable basis, this norm becomes Euclidean. Since in our case $\rho_2 = 1$, we have $\left(\frac{1}{m} \sum_{i=1}^m \|A_i z\|_2^2\right)^{1/2} = \|z\|_2$, $z \in \mathbb{R}^d$. Iterating k times, we get $\left(\frac{1}{m^k} \sum_{i_1, \dots, i_k} \|A_{i_k} \dots A_{i_1} z\|_2^2\right)^{1/2} = \|z\|_2$. Applying now the inequality between the quadratic and arithmetic mean we obtain

$$f_k(z) = \frac{1}{m^k} \sum_{i_1, \dots, i_k} \|A_{i_k} \dots A_{i_1} z\|_2 \leq \|z\|_2, \quad z \in \mathbb{R}^d.$$

This inequality becomes equality only if all the norms $\|A_{i_k} \dots A_{i_1} z\|_2$ are equal to $\|z\|_2$. Otherwise this inequality is strict. If it is strict for all $z \in \mathbf{S}^{d-1}$, then there is $\gamma < 1$ such that $f_k(z) \leq \gamma \|z\|_2$, $z \in \mathbb{R}^d$, which yields $\rho_1(\mathcal{A}) \leq \gamma^{1/k} < 1$. This contradiction shows that for each natural k , there is $z_k \in \mathbf{S}^{d-1}$ such that $\|A_{i_k} \dots A_{i_1} z_k\|_2 = \|z_k\|_2 = 1$ for all $i_1, \dots, i_k \in \{1, \dots, m\}$. Note that by submultiplicativity of operator norm the function $f_k(z)$ is non-increasing in k . Therefore, $f_k(z_k) = 1$ implies that $f_n(z_k) = 1$ for all $n \leq k$, and hence $\|A_{i_n} \dots A_{i_1} z_k\|_2 = 1$ for all $i_1, \dots, i_n \in \{1, \dots, m\}$. Consequently, for every limit point x of the sequence $\{z_k\}_{k \in \mathbb{N}}$ we have $\|Ax\|_2 = 1$, $A \in \mathcal{S}$. □

Remark 3 Theorem 3 asserts only the existence of a point x such that all its images Ax , $A \in \mathcal{A}$, lie on the surface of some ellipsoid. Thus, the point x makes walks on that ellipsoid under the action of the semigroup \mathcal{S} . If all matrices of \mathcal{S} are nonsingular, then all points $x \in \mathbb{R}^d \setminus \{0\}$ possess this property and, moreover, the ellipsoids are homothetic for all points. (Theorem 2). In general, however, this is not the case. In Remark 6 in Section 7 we shall see that the quadratic forms of all those ellipsoids are points of a common invariant affine plane of the tensor squares of matrices from \mathcal{S} . This gives a way to find those ellipsoids explicitly, for a given finitely generated semigroup $\mathcal{S}_\mathcal{A}$. As for the points x , we do not have a method to find at least one of them.

Remark 4 Theorem 3 holds only for Euclidean sphere, not for any other sphere of Banach norm in \mathbb{R}^d . To see this, it suffices to consider the irreducible c.s.r. semigroup $O(d)$.

Corollary 3 *An irreducible semigroup \mathcal{S} is c.s.r. if and only if there is a compact set $\Omega \in \mathbf{S}^{d-1}$ such that $\Omega = \cup_{A \in \mathcal{S}} A\Omega$.*

Proof . The sufficiency follows directly from Theorem 3. To prove the necessity we consider the set $\Omega_0 = \{Ax \mid A \in \mathcal{S}\}$, where $x \in \mathbf{S}^{d-1}$ is a point such that $Ax \in \mathbf{S}^{d-1}$ for all $A \in \mathcal{S}$. Let Ω_k be the closure of the set $\cup_{A \in \mathcal{S}} A\Omega_{k-1}$, $k \in \mathbb{N}$. Then $\Omega_k \subset \Omega_{k-1}$, $k \in \mathbb{N}$. The set $\Omega = \cup_{k \geq 0} \Omega_k$ possesses the desired property. □

6. Nonnegative c.s.r. semigroups

If all matrices in a semigroup are nonnegative (have nonnegative entries), the c.s.r. property can be efficiently characterized. The irreducibility assumption in this case is relaxed to the *positive irreducibility* (Definition 4 below).

We call a *coordinate subspace* a subspace of \mathbb{R}^d spanned by p basis vectors, where $1 \leq p \leq d-1$. A nonnegative matrix A is called *positively reducible*, if it has an invariant coordinate subspace. In this case there is a permutation of the canonical basis, after which A gets a block upper-triangular form with two blocks $p \times p$ and $(d-p) \times (d-p)$ on the diagonal, where p is the dimension of the common invariant coordinate subspace. Otherwise, if A has no common invariant coordinate subspaces, it is referred to as *positively irreducible*. In this case, for every i and j , there is a power $k \geq 1$ such that $(A^k)_{ij} > 0$.

Definition 4 *A family \mathcal{A} of nonnegative matrices is called positively reducible if all its matrices share a common invariant coordinate subspace. Otherwise \mathcal{A} is called positively irreducible.*

If \mathcal{A} is positively reducible, then there is a permutation of the canonical basis after which all matrices from \mathcal{A} get the block upper-triangular form (5) with square diagonal blocks of sizes $d^{(i)}$, $i = 1, \dots, s$, $\sum_{i=1}^s d^{(i)} = d$. The locations and sizes of the diagonal blocks are the same for all matrices $A \in \mathcal{A}$, the families $\mathcal{A}^{(i)}$ in the blocks are all positively irreducible. For positively irreducible families we still use factorization (5) with $s = 1$, $d^{(1)} = d$, $\mathcal{A}^{(1)} = \mathcal{A}$. If $\mathcal{A} = \mathcal{S}$ is a semigroup, then we denote by $\mathcal{S}^{(i)}$ the corresponding semigroups in the blocks. The following proposition and theorem almost repeat Proposition 2 and Theorem 1 respectively, but for nonnegative semigroups and for positive irreducibility. Their proofs are literally the same, replacing Barabanov's theorem on invariant norms for irreducible families by analogous [10, Theorem 3] for positively irreducible families.

Proposition 4 *A positively irreducible matrix semigroup has a bounded spectrum if and only if it is bounded.*

Theorem 4 *A nonnegative matrix semigroup \mathcal{S} is c.s.r. if and only if, in factorization (5), one of the positively irreducible semigroups $\mathcal{S}^{(j)}$ is c.s.r., and the others satisfy $\rho(\mathcal{S}^{(i)}) \leq 1$.*

The following criterion ensures the c.s.r. property of a positively irreducible semigroup.

Theorem 5 *A positively irreducible matrix semigroup \mathcal{S} is c.s.r. if and only if there is a proper affine subspace $V \subset \mathbb{R}^d$ such that $0 \notin V$, $V \cap \mathbb{R}_+^d \neq \emptyset$ and $AV \subset V$, $A \in \mathcal{S}$.*

The proof is in Appendix. Thus, an irreducible semigroup is c.s.r. precisely when it has an invariant affine plane (maybe one-point) intersecting the positive orthant and not passing through the origin. The following proposition ensures that every such a plane intersects an interior of positive orthant by a bounded set.

Proposition 5 *Under the assumptions of Theorem 5, the set $\mathbf{P} = V \cap \mathbb{R}_+^d$ is bounded (i.e., is a polyhedron) and intersects the interior of \mathbb{R}_+^d .*

The proof is in Appendix. Theorem 5 and Proposition 5 give a simple classification of nonnegative c.s.r. semigroups. One takes an arbitrary affine plane V of dimension from 0 to $d - 1$ that does not pass through the origin and intersects the positive orthant by a bounded set. Then every semigroup of nonnegative matrices respecting this plane has constant spectral radius, and vice versa: every positively irreducible c.s.r. semigroup is obtained this way. Note that the problem of classification of arbitrary c.s.r. semigroups (without the nonnegativity assumption) is much more difficult (Section 8).

Example 2 If \mathcal{S} is a semigroup of row-stochastic matrices, then $L = \mathbf{P} = \{e\}$, where $e = (1, \dots, 1)^T \in \mathbb{R}^d$. So, in this case $\dim L = 0$, although L is nonempty, and the polyhedron \mathbf{P} is one point. If \mathcal{S} is a semigroup of column-stochastic matrices, then $V = \{x \in \mathbb{R}^d \mid (e, x) = 1\}$ and \mathbf{P} is a simplex with vertices e_1, \dots, e_d .

Corollary 4 *If a semigroup of nonnegative matrices is c.s.r., then it is reducible.*

Proof. If \mathcal{S} is irreducible, then it is positively irreducible as well, and Theorem 4 gives a common affine invariant subspace L of matrices from \mathcal{S} such that $0 \notin L$. If $\dim L \geq 1$, then the linear part of L is a common invariant subspace for \mathcal{S} , if $\dim L = 0$, then so is the linear span of $\{0\} \cup L$. The contradiction completes the proof. \square

Let us recall that Corollary 4 deals with usual reducibility, not positive one. Thus, if nonnegative matrices generate a c.s.r. semigroup, then they share a common invariant linear subspace. Irreducible semigroups of nonnegative matrices are never c.s.r. For arbitrary semigroups, not necessarily nonnegative, this is not true. Examples can be found in [24] and in our Section 8.

Observe that if several matrices have a common affine subspace V , then so does every their convex combination. Invoking Theorem 5 we obtain: if a positively irreducible semigroup \mathcal{S} is c.s.r., then $\text{co}(\mathcal{S})$ also is. Theorem 4 extends this result to arbitrary nonnegative matrix semigroup. This proves the following

Theorem 6 *If \mathcal{S} is a nonnegative c.s.r. semigroup, then its convex hull $\text{co}(\mathcal{S})$ also is.*

The polyhedron $\mathbf{P} = V \cap \mathbb{R}_+^d$ is invariant with respect to each matrix $A \in \text{co}(\mathcal{S})$, i.e., $A\mathbf{P} \subset \mathbf{P}$. Applying the Brouwer fixed point theorem, we see that there exists $z \in \mathbf{P}$ such that $Az = z$. Since $\rho(A) = 1$, it follows that z is a Perron-Frobenius eigenvector for A .

Corollary 5 *Let \mathcal{S} be a positively irreducible c.s.r. and V be its common invariant affine subspace from Theorem 4. Then the polyhedron $\mathbf{P} = V \cap \mathbb{R}_+^d$ contains a Perron-Frobenius eigenvector of each matrix $A \in \text{co}(\mathcal{S})$.*

The following proposition gives a method to decide the c.s.r. property for a given nonnegative semigroup.

Proposition 6 *Suppose \mathcal{S} is a nonnegative matrix semigroup, $\bar{A} \in \text{co}(\mathcal{S})$ is a positively irreducible matrix, $\rho(\bar{A}) = 1$, and v is its Perron-Frobenius eigenvector; then \mathcal{S} is c.s.r. if and only if its matrices have a common invariant linear subspace that contains all vectors $v - Av$, $A \in \mathcal{S}$, and does not contain v .*

Proof . If there is a positively irreducible matrix $\bar{A} \in \text{co}(\mathcal{S})$, then \mathcal{S} is positively irreducible, the Perron-Frobenius eigenvector v of \bar{A} is positive and unique up to normalization. If there is a common invariant linear subspace \tilde{V} that contains all vectors $v - Av$, $A \in \mathcal{S}$ and does not contain v , then $V = v + \tilde{V}$ is a common invariant affine subspace for \mathcal{S} , and $0 \notin L$. Moreover, $v \in V \cap \mathbb{R}_+^d$, hence $V \cap \mathbb{R}_+^d \neq \emptyset$. Theorem 5 now yields that \mathcal{S} is c.s.r. Conversely, if \mathcal{S} is c.s.r., then so is $\text{co}(\mathcal{S})$ (Theorem 6), hence $\text{co}(\mathcal{S})$ possesses a common invariant affine subspace V , and (Corollary 5) the set $\mathbf{P} = V \cap \mathbb{R}_+^d$ contains the Perron-Frobenius eigenvector v of \bar{A} . Therefore, \tilde{V} is a common invariant linear subspace of \mathcal{S} containing all vectors $v - Av$, $A \in \mathcal{S}$. □

Remark 5 All the results of this section can be generalized from nonnegative matrices to matrices that share a common invariant cone K . This is an arbitrary convex solid pointed cone $K \subset \mathbb{R}^d$. In this case, the positive reducibility of \mathcal{S} means that all matrices from \mathcal{S} share a common invariant subspace which is a linear span of a face of K . Theorems 5 and 6, Propositions 5 and 6, and Corollaries 4, 5 remain true, if we replace \mathbb{R}^d by K , and nonnegative matrices by matrices leaving K invariant. In particular, Corollary 4 implies

Proposition 7 *None of irreducible c.s.r. semigroups possess a common invariant cone.*

Let us remark that for irreducible matrix semigroups, the equality $\rho_1(\mathcal{S}) = \rho(\bar{A})$, where $\bar{A} = \frac{1}{m} \sum_{i=1}^m A_i$, and A_1, \dots, A_m are matrices generating \mathcal{S} , is necessary and sufficient for the existence of a common invariant cone for K (see [30]).

7. How to verify the c.s.r. property ?

Given a finite family of matrices \mathcal{A} , how to decide whether the semigroup $\mathcal{S}_{\mathcal{A}}$ is c.s.r. ? Rather surprisingly, the answer is different for reducible and for irreducible families. In the irreducible case, the c.s.r. property can be verified by an efficient polynomial time algorithm, while in the reducible case, this problem is algorithmically undecidable even for a pair of nonnegative matrices. This means that there is no algorithm that for any pair of nonnegative rational matrices $\mathcal{A} = \{A_1, A_2\}$, decides this property within finite time. We start with the case of positively irreducible semigroups, when this problem is easily solvable.

7.1. Positively irreducible semigroups

To decide, whether a given positively irreducible semigroup \mathcal{S} is c.s.r. it suffices to take a positively irreducible matrix $\bar{A} \in \text{co}(\mathcal{S})$, which always exists, take its unique, up to normalization, Perron-Frobenius eigenvector v and consider the smallest (by inclusion) common invariant linear subspace \tilde{V} that contains all the vectors $v - Av$, $A \in \mathcal{S}$. Then \mathcal{S} is c.s.r. if and only if $v \notin \tilde{V}$.

If \mathcal{S} is generated by a finite family $\mathcal{A} = \{A_1, \dots, A_m\}$, then one can take $\bar{A} = \frac{1}{m} \sum_{i=1}^m A_i$ and \tilde{V} is the smallest common invariant subspace of \mathcal{A} that contains the vectors $v - A_i v$, $i = 1, \dots, m$. To find \tilde{V} , we denote by \tilde{V}_1 the linear span of those m vectors and iteratively construct the sequence $\tilde{V}_1 \subset \tilde{V}_2 \subset \dots$ as follows: \tilde{V}_{k+1} is the linear span of \tilde{V}_k and of $A_i \tilde{V}_k$, $i = 1, \dots, m$. For the smallest $k \leq d - 1$ such that $\tilde{V}_k = \tilde{V}_{k+1}$, the subspace $\tilde{V} = \tilde{V}_k$ is desirable. The algorithm performs at most $d - 1$ iterations, each consists of verifying whether a given new vector $A_i b_s$ belongs to the current subspace or has to be added to the basis (b_1, \dots, b_r is a basis of a current subspace, $r \leq d - 1$). Each verification is by computing of the corresponding determinant, and this has to be done for each $i = 1, \dots, m$ and $s = 1, \dots, r$. Thus, at most $d - 1$ iterations, each involves at most $m(d - 1)$ determinant computations in dimensions at most d . We see that the algorithm is polynomial.

The case of general irreducible semigroups (not necessarily nonnegative) is harder. Nevertheless, at least two effective methods exist for this case.

7.2. Irreducible semigroups. The first method

The algorithm for nonnegative semigroups can be extended directly to arbitrary irreducible semigroups by the Kronecker lifting. To any $d \times d$ -matrix A , one associates a linear operator $A^{\otimes 2} = A \otimes A$ on the space \mathcal{M}_d of symmetric $d \times d$ -matrices defined as follows: $A^{\otimes 2} X = A^T X A$, $X \in \mathcal{M}_d$. This is nothing else but the tensor product of the matrix A with itself. We have $\rho(A^{\otimes 2}) = \rho^2(A)$. Moreover, $A^{\otimes 2}$ leaves the cone \mathcal{K}_d of positively semidefinite matrices invariant. Thus, the Kronecker lifting transfers any family \mathcal{A} of $d \times d$ -matrices to the family $\mathcal{A}^{\otimes 2} = \{A^{\otimes 2} \mid A \in \mathcal{A}\}$ of linear operators over the $(d^2 + d)/2$ -dimensional space \mathcal{M}_d with a common invariant cone \mathcal{K}_d , and the same is for semigroups $\mathcal{S} \rightarrow \mathcal{S}^{\otimes 2}$. By increasing the dimension, we obtain a semigroup with a common invariant cone and with the same spectral properties. In particular, \mathcal{S} is c.s.r. if and only if $\mathcal{S}^{\otimes 2}$ is. Moreover, \mathcal{S} is irreducible if and only if $\mathcal{S}^{\times 2}$ is positively irreducible with respect to the cone \mathcal{K}_d , i.e., none faces of \mathcal{K}_d is a common invariant subspace for $\mathcal{S}^{\otimes 2}$. Therefore, we can verify the c.s.r. property of $\mathcal{S}^{\otimes 2}$ by the same algorithm as for nonnegative matrices, using an analog of Proposition 6 for the cone \mathcal{K}_d instead of \mathbb{R}_+^d (Remark 5). This is the idea of the first method of deciding the c.s.r. for irreducible family.

The algorithm. Given an irreducible semigroup \mathcal{S} , we need to decide whether it has the constant spectral radius or not. We take a positively irreducible operator $\hat{A} \in \text{co}(\mathcal{S}^{\otimes 2})$, take its Perron-Frobenius eigenvector $Y \in \mathcal{K}_d$ and consider the smallest (by inclusion) common invariant linear subspace $\tilde{V} \subset \mathcal{M}_d$ that contains all the matrices $Y - A^T Y A$, $A \in \mathcal{S}$. Then \mathcal{S} is c.s.r. if and only if $Y \notin \tilde{V}$. *End.*

If \mathcal{S} is generated by a finite family $\mathcal{A} = \{A_i\}_{i=1}^m$, then one can take $\hat{A} = \frac{1}{m} \sum_{i=1}^m A_i^{\otimes 2}$ and $\tilde{V} \subset \mathcal{M}_d$ is the smallest common invariant subspace of $\mathcal{A}^{\otimes 2}$ that contains the matrices $Y - A_i^T Y A_i$, $i = 1, \dots, m$. The construction of the subspace \tilde{V} is the same as for semigroups of nonnegative matrices, the complexity estimate is also the same, with replacement of the dimension from d to $(d^2 + d)/2$. Thus, if all the matrices A_i are rational, then the subspace \tilde{V} is found and the relation $Y \notin \tilde{V}$ is verified within polynomial time in the input. This proves the following theorem:

Theorem 7 *The c.s.r. recognition problem is solved by a polynomial time algorithm for an irreducible family of rational matrices, as well as for a positively irreducible family of nonnegative rational matrices.*

Remark 6 In fact, the common affine invariant subspace V for all operators $A^{\otimes 2}$, $A \in \mathcal{S}$ is closely related to walks on the Euclidean sphere from Section 5. By Theorem 3, an irreducible semigroup \mathcal{S} is c.s.r. precisely when there is a point $x \in \mathbb{R}^d$ and a matrix $H \in \mathcal{K}_d$ such that $x^T H x = 1$ and $x^T (A^T H A) x = 1$ for all $A \in \mathcal{S}$. Indeed, after linear change of coordinates taking the ellipsoid $\{y \in \mathbb{R}^d \mid y^T H y = 1\}$ to the unit Euclidean ball, these assertions become $\|x\|_2 = 1$, $\|Ax\|_2 = 1$, $A \in \mathcal{S}$. Hence, the affine hull V of all the matrices $A^T H A$, $A \in \mathcal{S}$, is invariant with respect to the operators of the semigroup $\mathcal{S}^{\otimes 2}$ and is contained in the affine hyperplane $V_x = \{M \in \mathcal{M}_d \mid x^T M x = 1\}$ of \mathcal{M}_d . Thus, V is defined by the orbit of the point $H \in \mathcal{M}_d$, which is the matrix of the ellipsoid, whose surface contains all the walk points Ax , $A \in \mathcal{S}$. Every point H from the intersection of V with the interior of \mathcal{K}_d is the quadratic form of such an ellipsoid, and all of them can be found by mere evaluating of the invariant subspace V . The point $x \in \mathbb{R}^d$, in turn, defines the desired affine hyperplane $V_x \subset \mathcal{M}_d$ which contains V . In practice, however, this is not clear how to find this point x algorithmically. The problem is apparently reduced to a system of quadratic equations of d variables. That is why, in the algorithm above we construct the invariant subspace V in a different way.

7.3. Irreducible semigroups. The second method

The method is based on the fact that the c.s.r. property of a semigroup \mathcal{S} generated by an irreducible finite family $\mathcal{A} = \{A_i\}_{i=1}^m$ is equivalent to the assertion $\rho_2(\mathcal{A}) = \rho_4(\mathcal{A}) = 1$, where ρ_p is the p -radius (Definition 3). The proof is actually the same as for the assertion $\rho_1(\mathcal{A}) = \rho_2(\mathcal{A}) = 1$ from the proof of Theorem 3. The values $\rho_2(\mathcal{A})$ and $\rho_4(\mathcal{A})$ are both efficiently computed: $\rho_2(\mathcal{A})$ is the square root of the spectral radius of the operator $\frac{1}{m} \sum_{i=1}^m A_i^{\otimes 2}$, and $\rho_4(\mathcal{A})$ is the power 1/4 of the spectral radius of the operator $\frac{1}{m} \sum_{i=1}^m A_i^{\otimes 4}$, where $A_i^{\otimes 4} = (A_i^{\otimes 2})^{\otimes 2}$ is the operator in the space of dimension $\binom{d}{4}$.

7.4. General semigroups. Undecidability

For reducible semigroups, however, the c.s.r. recognition problem is harder. This is explained by Theorem 1: to verify the c.s.r. property one needs to check that the joint spectral radii of the blocks $\mathcal{S}^{(i)}$ in factorization (5) do not exceed one. However, the problem of verifying the inequality $\rho(\mathcal{A}) \leq 1$ is algorithmically undecidable even for a pair of nonnegative rational matrices \mathcal{A} (see [5, 33]).

Theorem 8 *The c.s.r. recognition problem for a pair of nonnegative rational matrices is algorithmically undecidable.*

Proof. To an arbitrary pair $\mathcal{A} = \{A_1, A_2\}$ of $d \times d$ nonnegative matrices we associate a pair $\mathcal{A}' = \{A'_1, A'_2\}$ of $(d+1) \times (d+1)$ matrices as follows: A'_i has two diagonal blocks, the first block is $\{1\}$ (one-dimensional) the second block is A_i (d -dimensional), all other entries of A'_i are zeros. By Theorem 4, the semigroup generated by the pair \mathcal{A}' has constant spectral radius if and only if $\rho(\mathcal{A}) \leq 1$. Since the latter assertion is algorithmically undecidable, the former also is. □

8. A classification of c.s.r. semigroups

Apart from the case of nonnegative matrices (Section 6), the problem of complete classification of c.s.r. semigroups is not solved. In the next subsection, we describe several classes of c.s.r. semigroups and leave an open question about the completeness of that list. Then we prove Lemma 2 on the structure of s.c.r. semigroups, using which we classify them in low dimensions: $d = 2, 3$. By Theorem 1 it suffices to consider the irreducible case only. In what follows in this section, all semigroups are assumed to be irreducible. The main parameter of the classification is $r(\mathcal{S})$, the smallest rank of matrices from \mathcal{S} . The case $r = d$ is simple: \mathcal{S} consists of orthogonal matrices (Theorem 2). The case $r = d - 1$ is also classified, in all other cases we do not have a complete answer for $d \geq 4$. This issue is discussed in Subsection 8.5.

8.1. The list of irreducible c.s.r. semigroups

We spot the following classes of c.s.r. semigroups:

1. *An arbitrary subgroup of $O(d)$.*
2. *An arbitrary semigroup of (k, n) -matrices.*

For an irreducible c.s.r. semigroup \mathcal{S}_k of $k \times k$ matrices, $k \geq 1$, and for an arbitrary $n \geq 2$, a (k, n) -matrix A is defined as a matrix of size kn that consists of n^2 square blocks of size k ; each of n block rows of A contains a unique nonzero block from \mathcal{S}_k , all other blocks in that row are zero. Thus, any (k, n) -matrix contains exactly n nonzero blocks of size k , each of them equals to some matrix from \mathcal{S}_k . Clearly, any product of (k, n) -matrices (corresponding to

the same semigroup \mathcal{S}_k) is again a (k, n) -matrix. Hence, the norms of products are bounded uniformly from above and from below. Therefore, every set of (k, n) -matrices generate a c.s.r. semigroup of matrices of size $d = kn$.

Thus, for every irreducible c.s.r. semigroup of $k \times k$ -matrices and for any $n \geq 2$, the transfer to (k, n) -matrices produces c.s.r. semigroups of dimension kn .

Example 3 An $(1, n)$ -matrix is a matrix of size n that has a unique nonzero element ± 1 in each row, all other elements are zeros. If we denote by $\mathcal{I}_n = \{x \in \mathbb{R}^n \mid x_i = \pm 1, i = 1, \dots, n\}$ the set of vertices of an n -dimensional cube, then any $(1, n)$ -matrix A satisfies $A\mathcal{I}_n \subset \mathcal{I}_n$. This is a characteristic property of $(1, n)$ -matrices.

3. An arbitrary semigroup of (\mathbf{k}, \mathbf{n}) -torsion matrices.

Let $s \geq 2, \mathbf{k} = (k_1, \dots, k_s) \in \mathbb{Z}_+^s, \mathbf{n} = (n_1, \dots, n_s) \in \mathbb{Z}_+^s$, where $k_i, n_i \geq 1$ for all i and $n_j \geq 2$ for at least one j . We define two matrices B, C of size $d = \sum_{i=1}^s k_i n_i$ as follows. The matrix C is block-diagonal, it contains s square diagonal blocks C_i of sizes $k_i n_i, i = 1, \dots, s$, all other entries of C are zero. The i th block C_i is also a block matrix, it is composed of n_i^2 square blocks of size k_i . Exactly one of n_i block columns of C_i is nonzero, denote its number by h_i . All the n_i blocks $Q_{i, h_i, 1}, \dots, Q_{i, h_i, n_i}$ of this column are some orthogonal $k_i \times k_i$ -matrices. Let $L_{ij} \subset \mathbb{R}^d$ be the k_i -dimensional subspace corresponding to the j th block column of C_i . Thus, $\mathbb{R}^d = \bigoplus_{i,j} L_{i,j}$ and, respectively, $x = \sum_{i,j} x_{ij}, x_{ij} \in L_{i,j}$. Consider the following subspace of \mathbb{R}^d :

$$L = \left\{ x \in \mathbb{R}^d \mid Q_{i, h_i, 1}^{-1} x_{i1} = \dots = Q_{i, h_i, n_i}^{-1} x_{in_i}, i = 1, \dots, s \right\}$$

This subspace contains the range of C . Finally, let B be an arbitrary orthogonal operator on L . Then $A = BC$ is a (\mathbf{k}, \mathbf{n}) -torsion matrix.

Denote the following subset of the unit sphere in \mathbb{R}^d :

$$\mathcal{I}_{\mathbf{k}, \mathbf{n}} = \left\{ x \in \mathbb{R}^d \mid \|x_{i1}\|_2 = \dots = \|x_{in_i}\|_2, i = 1, \dots, s, \|x\|_2 = 1 \right\}.$$

For any (\mathbf{k}, \mathbf{n}) -torsion matrix A (the vectors \mathbf{k}, \mathbf{n} are fixed, the indices $h_i \in \{1, \dots, n_i\}$ and the orthogonal matrices $Q_{i, h_i, j}$ are arbitrary), we have $A\mathcal{I}_{\mathbf{k}, \mathbf{n}} \subset \mathcal{I}_{\mathbf{k}, \mathbf{n}}$. Hence, all products of (\mathbf{k}, \mathbf{n}) -torsion matrices define walks on the Euclidean sphere, and hence, each set of those matrices defines a c.s.r. semigroup.

4. An arbitrary semigroup generated by tensor products $A_1 \otimes A_2$, where $A_i \in \mathcal{S}_i$, and \mathcal{S}_i is an irreducible c.s.r. semigroup of matrices of size $d_i \geq 2, i = 1, 2$.

Thus, having a pair of irreducible c.s.r. semigroups $\mathcal{S}_1, \mathcal{S}_2$ of dimensions d_1 and d_2 respectively, we take an arbitrary set of tensor products of their matrices. This set generates a c.s.r. semigroup in dimension $d_1 d_2$.

5. The set of transpose matrices to a c.s.r. semigroup.

In particular, the semigroups of (k, n) -matrices from the class 2 defined by columns instead of rows (there is a unique nonzero block in each block column) is also c.s.r.

Of course, the linear similarity transform (a transfer to another basis) respects the c.s.r. property. We do not take it into account and consider all the classes 1 – 5 up to a linear similarity.

Problem 1. *Is it true that every irreducible c.s.r. semigroup is obtained from a subgroup of $O(d)$ (item 1) by the four procedures from items 2 – 5 and by linear similarities ?*

In Subsections 8.3 and 8.4, we shall see that the answer is affirmative for $d = 2, 3$ and for some other special cases. To attack the low-dimensional cases, we prove the following lemma which gives the main tool for c.s.r. classification.

8.2. The key lemma

We write $r(\mathcal{S})$ for the *rank of the semigroup \mathcal{S}* , i.e., the minimal rank of matrices from \mathcal{S} . Since \mathcal{S} is of constant spectral radius, it follows that $r \geq 1$. Theorem 2 implies that if $r = d$ for an irreducible semigroup \mathcal{S} , then it has a constant spectral radius if and only if \mathcal{S} consists of orthogonal matrices (in a suitable basis). Let us note that if \mathcal{S} is irreducible and c.s.r., then its rank coincides with the rank of its closure. Indeed, every matrix from \mathcal{S} has eigenvalues either zeros or of modulo one [24], hence taking a limit of such matrices does not reduce the rank. Without loss of generality we assume below that \mathcal{S} is closed.

Lemma 1 *For any irreducible c.s.r. semigroup \mathcal{S} , there is a subspace $L \subset \mathbb{R}^d$, $\dim L = r$, and a basis in \mathbb{R}^d such that the orthogonal projector P onto L belongs to \mathcal{S} and for every $A \in \mathcal{S}$, the operator $PA|_L$ is orthogonal.*

The proof is in Appendix.

Corollary 6 *For every irreducible c.s.r. semigroup \mathcal{S} there is basis in \mathbb{R}^d in which every matrix from \mathcal{S} has its upper principal $r \times r$ submatrix (in the first r rows and first r columns) orthogonal.*

Lemma 1 has the following geometrical meaning. If \mathbf{B}_r is an r -dimensional Euclidean ball in L , then $PA(\mathbf{B}_r) = \mathbf{B}_r$ for every $A \in \mathcal{S}$. In other words, every operator from \mathcal{S} maps the ball \mathbf{B}_r to an r -dimensional ellipsoid, which is a cross-section of the right circular cylinder $\mathbf{C}_r = \{x + y \mid x \in \mathbf{B}_r, y \perp L\}$ by some r -dimensional subspace of \mathbb{R}^d .

8.3. Low dimensions: $d = 2$.

If $r = 2$, then by Theorem 2 all matrices of \mathcal{S} are orthogonal. If $r = 1$, then by Lemma 1 there is an orthogonal projector P onto some one-dimensional subspace L such that for every $A \in \mathcal{S}$ the operator PA is orthogonal on L . Denote by $\mathbf{B} = [b_1, b_2]$ the segment of the line L of length 2 centered at the origin. We have $PAB = \mathbf{B}$ for each $A \in \mathcal{S}$, hence AB is a

segment centered at the origin with ends on the lines J_1 and J_2 that are orthogonal to \mathbf{B} and pass through the points b_1 and b_2 respectively. Thus, all operators $A \in \mathcal{S}$ map \mathbf{B} to such segments. Let us show that the set $\{A\mathbf{B} \mid A \in \mathcal{S}\}$ actually consists of two segments, including \mathbf{B} . By Lemma 1, $P \in \mathcal{S}$, hence the segment \mathbf{B} belongs to this set. If all segments of this set coincide with \mathbf{B} , then \mathcal{S} is reducible. Hence, there is a segment $[c_1, c_2] \neq [b_1, b_2]$ in this set. Assume there is a third segment $[d_1, d_2]$, where $c_i, d_i \in L_i$, $i = 1, 2$. If for all $A \in \mathcal{S}$ the points Ac_1 and Ad_1 lie on the same line J_i then $(AJ_1) \perp L$ for all $A \in \mathcal{S}$, hence \mathcal{S} has a common invariant subspace parallel to J_1 . Therefore, for some $A \in \mathcal{S}$ the points Ac_1 and Ad_1 lie on two different lines J_i . However, in this case the point Ab_1 does not lie on these lines (since Ab_1, Ac_1 and Ad_1 are co-linear), which is impossible. Thus, the set of all images of \mathbf{B} consists of exactly two segments. Every operator $A \in \mathcal{S}$ maps each of these segments either to itself or to the other one. After a suitable linear transform, we assume that these segments are two diagonals of a unit square \mathbf{Q} . Then every operator from \mathcal{S} is either a projection to one of diagonal of \mathbf{Q} parallel to one of its sides, or an isometry of \mathbf{Q} , or composition of several such operators. Thus, we have proved

Proposition 8 *All irreducible c.s.r. semigroups \mathcal{S} of 2×2 -matrices are classified as follows:*

if $r = 2$, then \mathcal{S} consists of orthogonal matrices;

if $r = 1$, then \mathcal{S} consists of compositions of the following operators:

1) projections on a diagonal of \mathbf{Q} parallel to its side (4 operators);

2) orthogonal transforms of \mathbf{Q} (two rotations on $\pi/2$, four axial symmetries, and the central symmetry).

If $\mathbf{Q} = \{x \in \mathbb{R}^2 \mid \max\{|x_1|, |x_2|\} \leq 1\}$, then the matrices from classes 1) and 2) in Proposition 8 are actually $(1, 2)$ -matrices (Example 3). Thus, in the case $d = 2, r = 1$, every irreducible c.s.r. semigroup consists of matrices with exactly one nonzero element ± 1 in each row.

Corollary 7 *If an irreducible semigroup of 2×2 -matrices of constant spectral radius contains at least one degenerate matrix, then it is finite and consists of $(1, 2)$ -matrices.*

8.4. Low dimensions: $d = 3$.

If $r = 3$, then all matrices of \mathcal{S} are orthogonal. Consider the other cases.

The case $r = 1$. By Lemma 1 there is an orthogonal projector P onto some one-dimensional subspace $L \subset \mathbb{R}^3$ such that for every $A \in \mathcal{S}$ the operator PA is orthogonal on L . Let $\mathbf{B} = [b_1, b_2]$ be the segment of L of length 2 centered at the origin. For each $A \in \mathcal{S}$, the segment $[x_1, x_2] = A\mathbf{B}$ is centered at the origin and $x_i \in J_i$, $i = 1, 2$ where J_1 and J_2 are two-dimensional planes orthogonal to \mathbf{B} and passing through the points b_1 and b_2 respectively. From the irreducibility of \mathcal{S} it follows that for some $C \in \mathcal{S}$ the corresponding segment $[c_1, c_2] = C\mathbf{B}$ does not coincide with $[b_1, b_2]$. Moreover, there is an operator $D \in \mathcal{S}$ that sends the points b_1 and c_1 to different planes L_i . Otherwise the linear span of the

vectors $\{A(c_1 - b_1) \mid A \in \mathcal{S}\}$ is a common nontrivial invariant subspace for \mathcal{S} . Take now arbitrary $A \in \mathcal{S}$ and consider the segment $[x_1, x_2] = A\mathbf{B}$. The point Dx_1 belongs to either J_1 or J_2 . The set $\{x \in J_1 \mid Dx \in J_1\}$ is either empty, or a line in J_1 , of the whole J_1 . Since it contains precisely one of the two points b_1 and c_1 , it is neither empty nor the whole J_1 , so it is a line. The set $\{x \in J_1 \mid Dx \in J_2\}$ is a parallel line. Thus, x_1 belongs to one of these parallel lines. Since this is true for every $A \in \mathcal{S}$, we see that the ends of all segments $A\mathbf{B}$, $A \in \mathcal{S}$ on the plane J_1 lie on this pair of parallel lines. By the irreducibility of \mathcal{S} , there is another pair of parallel lines, which is not parallel to the first pair, that also contains all these ends of the segments $A\mathbf{B}$, $A \in \mathcal{S}$. Whence, all these ends are located at common points of these four lines, i.e., at four vertices of a parallelogram. Therefore, all the segments $A\mathbf{B}$, $A \in \mathcal{S}$ are diagonals of the parallelepiped \mathbf{P} centered at the origin, whose face coincide with that parallelogram. After a suitable linear transform, it may be assumed that \mathbf{P} is a unit cube, whose vertices have coordinates ± 1 . If a matrix $A \in \mathcal{S}$ maps each diagonal of \mathbf{P} to a diagonal, then A is a $(1, 3)$ -matrix (Example 3). If A respects the set of three diagonals, then, in the basis of these diagonals, A is a transpose to a $(1, 3)$ -matrix. Thus, we have proved

Proposition 9 *Up to a linear similarity, every irreducible c.s.r. semigroup of 3×3 -matrices for which $r = 1$, either consists of $(1, 3)$ -matrices, or consists of transposes to $(1, 3)$ -matrices.*

Thus, in case $d = 3, r = 1$, every irreducible c.s.r. semigroup either consists of matrices with a unique nonzero element ± 1 in each row, or consists of matrices with a unique nonzero element ± 1 in each row.

The case $r = 2$. By Lemma 1 there is an orthogonal projector P onto a two-dimensional subspace $L \subset \mathbb{R}^3$ such that for every $A \in \mathcal{S}$ the operator PA is orthogonal on L . Let \mathbf{B} be the Euclidean unit ball in L . For each $A \in \mathcal{S}$, the set $A\mathbf{B}$ is a cross-section of a right circular cylinder $\mathbf{C} = \{x + y \mid x \in \mathbf{B}, y \perp L\}$ by some two-dimensional subspace of \mathbb{R}^3 . Assume there are operators $A_1, A_2 \in \mathcal{S}$ such that the three sets $A_1\mathbf{B}, A_2\mathbf{B}$, and $A\mathbf{B}$ are all different. Let a line orthogonal to the plane \mathbf{B} on the surface of the cylinder meets these cross-sections at points a_1, a_2 , and a respectively. From the irreducibility of \mathcal{S} it follows that there is an operator $D \in \mathcal{S}$ such that the line passing through Da_1 and Da_2 is not orthogonal to L . Hence, this line intersects the surface of the cylinder \mathbf{P} at most at two points, which is impossible, because the three points Da_1, Da_2, Da belongs to that intersection. Thus, the set $\{A\mathbf{B} \mid A \in \mathcal{S}\}$ consists of two cross-sections of the cylinder, one of which is \mathbf{B} . Up to an affine similarity, it may be assumed that these two cross-sections are equal concentric Euclidean discs orthogonal to each other. The semigroup \mathcal{S} consists of operators that map each of these discs to itself or to another disc.

Proposition 10 *Up to a linear similarity, every irreducible c.s.r. semigroup of 3×3 -matrices for which $r = 2$, consists of operators that map each of the two equal concentric orthogonal Euclidean discs to itself or to the other disc.*

Such semigroup may contain the following operators and their compositions:

1) a composition of the projection of the first disc onto the second one with an orthogonal transform of the second disc;

2) orthogonal transforms respecting the discs: rotation on $\pi/2$ and on π around the common line of the discs, symmetries with respect to the planes of the discs and with respect to their bisector planes, central symmetry with respect to the origin. If the planes of the disc have the equations $L_{\pm} = \{x \in \mathbb{R}^3 \mid x_1 = \pm x_2\}$, then the operators of the class 1) are given by (\mathbf{k}, \mathbf{n}) -torsion matrices with $\mathbf{k} = (1, 1)$, $\mathbf{n} = (2, 1)$ (item 3 in the list of Subsection 8.1). Each of them is the product $A = BC$, where C consists of two diagonal blocks: C_1 is a 2×2 matrix with one nonzero column with entries ± 1 ; C_2 is ± 1 . The matrix B defines an orthogonal transform of the two-dimensional plane L_+ or L_- (the plane containing the disc).

The operators of class 2) are given by $(1, 3)$ -matrices (Example 3).

8.5. Arbitrary dimension d

Already for $d = 4$ we are not able to classify all irreducible c.s.r. semigroups. Using items 1 – 5 in Subsection 8.1 we obtain the following c.s.r. families of 4×4 -matrices:

- an arbitrary subset of the group $O(4)$;
- an arbitrary set of $(1, 4)$ -matrices; a transpose to that set;
- a set of $(2, 2)$ -matrices made of an arbitrary c.s.r. family of matrices of size 2, a transpose to that set;
- an arbitrary set of tensor products of two c.s.r. families of 2×2 matrices (Subsection 8.3);

an arbitrary set of (\mathbf{k}, \mathbf{n}) -torsion matrices (item 3, Subsection 8.1) either with $\mathbf{k} = (1, 1)$, $\mathbf{n} = (3, 1)$, or with $\mathbf{k} = (1, 1)$, $\mathbf{n} = (2, 2)$, or with $\mathbf{k} = (1, 2)$, $\mathbf{n} = (2, 1)$, or with $\mathbf{k} = (1, 1, 1)$, $\mathbf{n} = (2, 1, 1)$.

Of course, all families obtained by those ones by a change of basis are also c.s.r. We do not know whether this list is complete.

For arbitrary d , the construction of c.s.r. semigroups is realized by items 1 – 5 of Subsection 8.1 in the same way, although the number of cases grows dramatically with the dimension. The problem of completeness of this classification is solved by now only for two values of r . If $r = d$, then, apparently, \mathcal{S} is a subgroup of $O(d)$. If $r = d - 1$, then a complete analogue of Proposition 10 takes place (with essentially the same proof): every irreducible c.s.r. semigroup of $d \times d$ -matrices for which $r = d - 1$, consists of operators that map each of the two equal concentric orthogonal Euclidean $(d - 1)$ -dimensional balls to itself or to the other ball. The class 1) of Proposition 10 is described now by (\mathbf{k}, \mathbf{n}) -torsion matrices with $\mathbf{k} = (1, \dots, 1)$, $\mathbf{n} = (2, 1, \dots, 1)$ (the vectors are both $(d - 1)$ -dimensional); the class 1) is by $(1, d)$ -matrices.

9. Applications

9.1. Finite matrix semigroups

Finite matrix semigroups have been studied in the literature, see [6, 13, 23, 22] and bibliography in those works. In fact, they are closely related to c.s.r. semigroups.

Proposition 11 *If a finite matrix semigroup \mathcal{S} does not contain zero matrix, then \mathcal{S} is c.s.r.*

Proof. If, to the contrary, $\rho(A) \neq 1$ for some $A \in \mathcal{S}$, then all numbers $\rho(A^k) = \rho^k(A)$ are different, whenever $\rho(A) \neq 0$, and therefore, \mathcal{S} contains an infinite subset $\{A^k, k \in \mathbb{N}\}$. If $\rho(A) = 0$, then A is nilpotent, and hence $A^d = 0$, which is impossible, since $A^d \in \mathcal{S}$. \square

Invoking Theorem 2, we obtain

Corollary 8 *For every irreducible finite matrix semigroup \mathcal{S} , there is a basis in \mathbb{R}^d in which all nonsingular elements of \mathcal{S} are orthogonal. Every irreducible finite semigroup of nonsingular matrices is, in a suitable basis, a subgroup of $O(d)$.*

Proof. The first part follows from Theorem 2. In the second part, it remains to prove that a finite semigroup \mathcal{S} of nonsingular matrices is a group, i.e., contains an inverse to each $A \in \mathcal{S}$. Indeed, since the sequence $\{A^k\}_{k \in \mathbb{N}} \subset \mathcal{S}$ is finite, it follows that $A^k = A^{k+n}$ for some $k, n \in \mathbb{N}$. Hence, $A^n = I$ and $A^{n-1} = A^{-1}$. Since $A^{n-1} \in \mathcal{S}$, the proof is completed. \square

Thus, all finite irreducible semigroups of nonsingular matrices are finite subgroups of $O(d)$. Note that every nonsingular element of a finite matrix semigroup is evidently similar to an orthogonal matrix, otherwise all its powers are different. What is nontrivial in Corollary 8 is that all nonsingular matrices become orthogonal simultaneously, in one basis.

Finite semigroups not containing zero matrices form a subclass of c.s.r. semigroups. For integer matrices, i.e., for matrices with integer entries, these classes coincide.

Proposition 12 *An irreducible c.s.r. semigroup of integer matrices is finite. The same holds for positively irreducible semigroups of nonnegative integer matrices.*

Proof. All norms of matrices in an irreducible c.s.r. semigroup are bounded by some constant, hence the number of integer matrices is finite. \square

The problem of classification of finite semigroups is still open, even for integer matrices. The results of Section 8 give such a classification in low dimensions ($d = 2, 3$) and many examples of finite semigroups in higher dimensions. Starting with a finite subgroup of $O(d)$ and applying procedures 2 – 5 from the list in Subsection 8.1 we obtain finite semigroups. As in Problem 1, one may rise the question about the completeness of this classification.

The algorithmic recognition of finiteness for the semigroup generated by a given family of matrices $\mathcal{A} = \{A_1, \dots, A_m\}$ is also a challenging problem. Algorithms presented in [22, 14] solve it for nonnegative integer matrices (the algorithm from [14] is polynomial). We make the next step and solve the problem for arbitrary irreducible family \mathcal{A} of integer matrices, provided it is not mortal, i.e., none of products of those matrices is zero. Note that the mortality recognition problem is known to be algorithmically undecidable [33].

Theorem 9 *There is a polynomial-time algorithm that for every irreducible non-mortal family of integer matrices $\mathcal{A} = \{A_1, \dots, A_m\}$ decides whether it generates a finite semigroup.*

Proof. In view of Propositions 11 and 12, the finiteness of $\mathcal{S}_{\mathcal{A}}$ is equivalent to its c.s.r. property, which can be decided by the algorithm from Subsection 7. (Theorem 7). \square

There are many examples of finite semigroups of integer matrices. We consider one class obtained by iterating the construction of (k, n) -matrices (item 3, Subsection 8.1).

Example 4 Take a finite sequence n_1, \dots, n_q , each n_i is natural and is not one. Consider an arbitrary set of $(1, n_1)$ -matrices (Example 3) and transpose them. We get a set of matrices of size n_1 , each of them has one nonzero element ± 1 in every column. Then, from this set, construct an arbitrary set of (n_1, n_2) -matrices (item 2 in the list in Subsection 8.1) and transpose them. Then construct a set of $(n_1 n_2, n_3)$ -matrices, transpose them, and so on. As a result, we obtain a c.s.r. family \mathcal{A} of matrices of size $d = n_1 \dots n_q$. All their entries are zeros and plus/minus ones, all their products are uniformly bounded, hence the semigroup $\mathcal{S}_{\mathcal{A}}$ is finite. Take now an arbitrary integer matrix C such that $|\det C| = 1$. Then the family $\{C^{-1}AC \mid A \in \mathcal{A}\}$ generates a finite semigroup of integer matrices.

9.2. Linear switching systems

Given a compact family \mathcal{A} of $d \times d$ matrices, a linear switching system (LSS) is the following linear differential equation on the vector-valued function $x : [0, +\infty) \rightarrow \mathbb{R}^d$:

$$\begin{cases} \dot{x}(t) = A(t)x(t); \\ x(0) = x_0, \end{cases} \quad (6)$$

where $A(\cdot) : [0, +\infty) \rightarrow \mathcal{A}$ is a measurable function called the *switching law*. The solution $x(\cdot)$ is a *trajectory* of the system corresponding to the switching law $A(\cdot)$ and to the initial condition $x(0) = x_0$. There is an extensive literature on LSS, in particular, exploring the asymptotic growth of the trajectories (see [17, 18, 21] and references therein). The *exponent of growth* of the switching law $A(\cdot)$ is

$$\sigma(A(\cdot)) = \sup_{x_0 \in \mathbb{R}^d} \left(\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|x(t)\| \right),$$

where $x(t)$ is the trajectory with the initial condition $x(0) = x_0$. This is the fastest growth of all trajectories corresponding to a given switching law. We call a system *uniform* if $\sigma(A(\cdot))$ is the same for all switching laws $A(\cdot)$. Our goal is to characterize uniform systems.

Observe that making an α -shift of the system: $\mathcal{A} \mapsto \mathcal{A} + \alpha I = \{A + \alpha I \mid A \in \mathcal{A}\}$, we replace every trajectory $x(t)$ by $e^{\alpha t}x(t)$, and, therefore, add α to every exponent of growth $\sigma(A(\cdot))$. Hence, up to a suitable shift, it may be assumed that $\sigma(A(\cdot)) = 0$, and it suffices to characterize only such uniform systems.

Proposition 13 *An LSS is uniform with $\sigma(A(\cdot)) = 0$ for every switching law, if and only if the family $\mathcal{M} = \{e^{tA} \mid A \in \mathcal{A}, t \in \mathbb{R}_+\}$ is c.s.r.*

Proof. If \mathcal{M} is not c.s.r., then there are numbers $t_i > 0$ and matrices $A_i \in \mathcal{A}, i = 1, \dots, n$, such that $\lambda = \rho(\prod_{i=1}^n e^{t_i A_i}) \neq 1$. Denote $T = \sum_{i=1}^n t_i$. For the piecewise-constant switching law $A(t)$ taking values A_n, \dots, A_1 on successive segments of lengths t_n, \dots, t_1 , we have $x(T) = e^{t_1 A_1} \dots e^{t_n A_n} x(0)$. Therefore, for the periodization of this switching law with period T , we have $\sigma(A(\cdot)) = T^{-1} \log \lambda \neq 0$, hence the LSS is not uniform. For the converse, in view of Theorem 1, it suffices to consider an irreducible family \mathcal{A} , in which case \mathcal{M} is also irreducible. If \mathcal{M} is c.s.r., then the norms of all products of its matrices is between two positive constants C_1 and C_2 . Hence, for any piecewise-constant switching law, we have

$$C_1 \leq \max_{\|x_0\|=1} \|x(t)\| \leq C_2.$$

By continuity, this holds for every switching law $A(\cdot)$, and hence $\sigma(A(\cdot)) = 0$. □

By Theorem 1, it suffices to characterize only irreducible uniform LSS (corresponding to irreducible families \mathcal{A}). Note that all matrices of \mathcal{M} are invertible, hence the c.s.r. property can be characterized by applying Theorem 2.

Theorem 10 *An irreducible LSS is uniform with $\sigma = 0$ if and only if there is a basis in \mathbb{R}^d in which all matrices of \mathcal{A} are antisymmetric ($A^T = -A$). In this case, every trajectory $x(t), t \in [0, +\infty)$, lies on a Euclidean sphere.*

Proof. If the LSS is uniform, then the family of nonsingular matrices \mathcal{M} is c.s.r., and by Theorem 2, all the matrices e^{tA} are orthogonal in a suitable basis. This means $A^T = -A$ for all $A \in \mathcal{A}$. Conversely, if all matrices of \mathcal{A} are antisymmetric, then $(\dot{x}(t), x(t)) = (A(t)x(t), x(t)) = 0$, and consequently $\|x(t)\|_2 \equiv \text{const}$ along every trajectory. □

An important class of LSS are *positive systems*, for which all trajectories are nonnegative ($x(t) \geq 0, t \in \mathbb{R}_+$), provided $x(0) \geq 0$. An LSS is positive precisely when each matrix $A \in \mathcal{A}$ is Metzler, i.e. all off-diagonal elements of A are nonnegative. About properties of positive LSS, see [8, 18] and references therein. To characterize positive uniform LSS we apply Theorem 5 on nonnegative c.s.r. semigroups.

Theorem 11 *A positively irreducible LSS is uniform with $\sigma = 0$ if and only if the family $I + \mathcal{A}$ has a proper affine invariant subspace $V \subset \mathbb{R}^d$ that intersects the positive orthant and does not contain the origin. In this case, every trajectory starting in the set $\mathbf{P} = V \cap \mathbb{R}_+^d$ never leaves \mathbf{P} .*

Proof. If the LSS is uniform, then the family \mathcal{M} is c.s.r. Since this is a positively irreducible family of nonnegative matrices, Theorem 5 implies the existence of an affine subspace V such that $0 \notin V$, the set $\mathbf{P} = V \cap \mathbb{R}_+^d$ is nonempty and compact, and $e^{tA}\mathbf{P} \subset \mathbf{P}$, $A \in \mathcal{A}$, $t \in \mathbb{R}_+$. The latter means that for every switching law $A(t)$, if $x(0) \in \mathbf{P}$, then $x(t) \in \mathbf{P}$ for all $t > 0$. Hence, $\dot{x}(t) \in \tilde{V}$, and therefore $A(t)x(t) \in \tilde{V}$ for all $t > 0$ (let us recall that \tilde{V} is the linear part of V). On the other hand, $x(t) \in V$, and so $(I + A(t))x(t) \in x(t) + \tilde{V} = V$. Thus, V is invariant for $I + \mathcal{A}$, and the trajectory $x(t)$, $t > 0$, lies in \mathbf{P} . Conversely, if the desired subspace V exists, then $AV \subset \tilde{V}$, $A \in \mathcal{A}$, hence, $\dot{x}(t) = A(t)x(t) \in \tilde{V}$, whenever $x(t) \in V$. Thus, every trajectory $x(t)$ starting in V stays in V . Therefore, $e^{tA}V \subset V$ for every $e^{tA} \in \mathcal{M}$, and, by Theorem 5, the family \mathcal{M} is c.s.r. \square

9.3. Regularity of fractal curves

Let $\{B_0, B_1\}$ be an irreducible pair of affine contractions in \mathbb{R}^d . The irreducibility means that they do not share a common *affine* invariant plane. Since B_i is a contraction, it has a unique fixed point: $B_i v_i = v_i$, $i = 0, 1$. Assume also the *Barnsley condition* (*cross-condition*): $B_0 v_1 = B_1 v_0$. Then the following functional equation:

$$v(t) = \begin{cases} B_0 v(2t) & , \quad t \in [0, \frac{1}{2}] ; \\ B_1 v(2t - 1) & , \quad t \in [\frac{1}{2}, 1] \end{cases} \quad (7)$$

possesses a unique continuous solution $v : [0, 1] \rightarrow \mathbb{R}^d$ called a *fractal curve* [3, 21]. We consider the simplest case of equation (7), with two operators and with double contraction of the argument. For more general constructions, see [9, 28, 36]. The Hölder exponent of the solution is $\alpha_v = -\log_2 \rho(B_0, B_1)$. The local regularity at each point $t \in [0, 1]$ is measured by the local Hölder exponent:

$$\alpha_v(t) = \sup \left\{ \alpha > 0 \mid |v(t+h) - v(t)| \leq C h^\alpha \right\}.$$

By [28, Theorem 5], for every point $t \in [0, 1]$, we have $\alpha_v(t) \in [\alpha_{\min}, \alpha_{\max}]$ with

$$\alpha_{\min} = -\log_2 \rho(B_0, B_1) ; \quad \alpha_{\max} = -\log_2 \check{\rho}(B_0, B_1),$$

where, let us remember, $\rho = \rho_{+\infty}$ is the joint spectral radius and $\check{\rho} = \check{\rho}_{-\infty}$ is the lower spectral radius (Section 2). Moreover, if B_0, B_1 are both nonsingular, then for any $\nu \in [\alpha_{\min}, \alpha_{\max}]$ the set of points t for which $\alpha_v(t) = \nu$ is everywhere dense in the segment $[0, 1]$. This set is of Lebesgue measure zero for all but one ν , for which it is of measure 1 (this is for $\nu_0 = -\log_2 \rho_0(B_0, B_1)$, where ρ_0 is the Lyapunov exponent).

Thus, every fractal curve has a varying local regularity. In each interval $(t_0, t_1) \in [0, 1]$, all values of the local Hölder exponent $\alpha_v(t)$ from α_{\min} to α_{\max} are attained. The only exception is when $\alpha_{\min} = \alpha_{\max}$, in which case the local regularity is the same at all points $t \in [0, 1]$. Our goal is to characterize fractal curves with constant local regularity.

The equality $\alpha_{\min} = \alpha_{\max}$ means that $\check{\rho}(\tilde{B}_0, \tilde{B}_1) = \rho(\tilde{B}_0, \tilde{B}_1)$, which is equivalent, in view of Proposition 1, to the existence of number $r > 0$ such that the pair $\{r^{-1}\tilde{B}_0, r^{-1}\tilde{B}_1\}$ is c.s.r. We use factorization (5) for matrices of the linear parts \tilde{B}_0, \tilde{B}_1 of operators B_0, B_1 . Applying Theorem 1 and 2, we obtain

Theorem 12 *A fractal curve generated by nonsingular operators B_0, B_1 has a constant local regularity if and only if there is a basis in \mathbb{R}^d in which both matrices \tilde{B}_0, \tilde{B}_1 have the form (5), where in one diagonal block, both matrices $\tilde{B}_0^{(j)}$ and $\tilde{B}_1^{(j)}$ are orthogonal multiplied by a number $r \in [\frac{1}{2}, 1)$, and in the other blocks, $\rho(B_0^{(i)}, B_1^{(i)}) \leq r$, $i \neq j$.*

In this case $\alpha_v(t) = -\log_2 r$ at all points $t \in [0, 1]$.

If the operators B_0, B_1 have irreducible linear parts, then the situation is much simpler:

Corollary 9 *If B_0, B_1 are nonsingular affine operators whose linear parts do not share a common invariant subspace, then the fractal curve has a constant local regularity if and only if these operators are similarities with the same contraction factor $r \in [\frac{1}{2}, 1)$.*

Example 5 The famous Koch curve (“the Koch snowflake”) has constant local regularity $\alpha_v(t) = \log_2 \sqrt{3}$, because it is generated by two similarities with $r = 1/\sqrt{3}$. Another famous fractal curve is De Rham curve obtained as a limit of the cutting angle algorithm from a polygon. In each iteration, all sides of a current polygon are divided into three parts with the same ratio $\omega : (1 - 2\omega) : \omega$, where $\omega \in (0, 1/2)$. This curve has a constant local regularity only for $\omega = \frac{1}{4}$, for other values of ω we have $\alpha_{\min} < \alpha_{\max}$ (see [28]). Other important examples are refinable functions and wavelets considered in the next subsection.

9.4. Refinable functions and wavelets

The following functional equation on a compactly supported scalar function φ plays a crucial role in the construction of compactly supported wavelets and of subdivision schemes in approximation theory, curve and surface design (see [21, 7, 28, 34] and references therein):

$$\varphi(x) = \sum_{k=0}^N c_k \varphi(2t - k). \quad (8)$$

This is called *refinement equation*. Its compactly supported solution φ (*refinable function*) is a fixed point of *transition operator* $[Tf](t) = \sum_{k=0}^N c_k f(2t - k)$. It can be assumed that $\sum_{k=0}^N c_k = 2$, the general case can always be reduced to this one. In this case, equation (8) has a unique, up to normalization, solution φ in the space of distributions. We normalize this solution by the condition $\int_{\mathbb{R}} \varphi(t) dt = 1$. This solution is supported on the segment $[0, N]$ and for an arbitrary compactly supported distribution f such that $\int_{\mathbb{R}} f(t) dt = 1$, one has $T^k f \rightarrow \varphi$ as $k \rightarrow \infty$ (the convergence is in the sense of distributions). The main problem is when this solution is summable or continuous and what is its regularity and other properties. A necessary condition for summability of φ are formulated in terms of the generating polynomial $m(z) = \frac{1}{2} \sum_{k=0}^N c_k z^k$.

Theorem A [34]. *If the solution of refinement equation (8) is summable, then at least one of the following two conditions are fulfilled: a) $m(-1) = 0$; b) there is $z \in \mathbb{C}$ such that $m(z) = m(-z) = 0$.*

Continuous refinable functions generate wavelet functions in construction of compactly supported wavelets (for instance, Daubechies wavelets), the limit function of a subdivision scheme, etc. The specific fractal-like properties of refinable functions are explained by the fact that equation (8) becomes the equation on fractal curve (7) for the vector-function $v(t) = v_\varphi(t) = (\varphi(t), \dots, \varphi(t - N + 1))^T$ with the operators B_0, B_1 given by $N \times N$ matrices:

$$(B_k)_{ij} = c_{2i-j+k-1}, \quad i, j = 1, \dots, N; \quad k = 0, 1 \quad (9)$$

(if the index $2i - j + k - 1$ is negative or exceeds N , then we set $c_{2i-j+k-1} = 0$). This equation is considered on the affine subspace $V \subset \mathbb{R}^N$, the smallest by inclusion common invariant affine subspace of the matrices B_0, B_1 passing through the point $v(0)$, which is an eigenvector of B_0 corresponding to the eigenvalue 1.

Thus, $v_\varphi(t)$ is a fractal curve of operators B_0, B_1 on V . In particular, it has a varying local regularity in the sense described in the previous subsection. The same property is inherited by all refinable functions $\varphi(t)$, wavelet functions, and limit functions of subdivision schemes. Our aim is to characterize refinable functions with the constant local regularity, i.e., the case $\alpha_{\min} = \alpha_{\max}$.

Conjecture 1 *If a refinable function has constant local Hölder exponent α , then $\alpha = 1$.*

Thus, either a refinable function φ is smooth (for instance, Lipschitz continuous), in which case $\alpha_\varphi(t) = 1$ for all t , or φ has varying local regularity and $\alpha_{\min} < \alpha_{\max}$. We are able to prove Conjecture 1 only under some mild assumption:

Proposition 14 *Conjecture 1 holds true unless all roots of $m(z)$ are on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.*

Proof. We use several results from [29]. There is a basis in the space V such that the matrices $B_i|_V$ have block lower triangular form with numbers $\frac{1}{2}, \dots, \frac{1}{2^q}$ and a block \hat{B}_i on the diagonal, $i = 0, 1$. The matrices \hat{B}_i are both nondegenerate and the pair $\{\hat{B}_0, \hat{B}_1\}$ is irreducible. These matrices have the same form (9) with a special sequence $\hat{c}_1, \dots, \hat{c}_n$; the generating polynomial of this sequence $\hat{m}(z) = \frac{1}{2} \sum_{k=0}^n \hat{c}_k z^k$ have all roots on the unit circle if and only if the polynomial $m(z)$ does. From the structure of the matrix $B_i|_V$ it follows that if $r = \rho(\hat{B}_0, \hat{B}_1) \leq \frac{1}{2}$, then $\check{\rho}(B_0|_V, B_1|_V) = \rho(B_0|_V, B_1|_V) = \frac{1}{2}$, hence $\alpha_{\min} = \alpha_{\max} = 1$, which completes the proof. If $r > \frac{1}{2}$, then $\alpha_{\min} = \alpha_{\max}$ if and only if the pair $\{r^{-1}\hat{B}_0, r^{-1}\hat{B}_1\}$ is c.s.r. Since this pair is irreducible and the matrices are nonsingular, it follows from Theorem 2 that both these matrices are orthogonal in some basis. Hence, the matrix $\hat{B}_0^{-1}\hat{B}_1$ is orthogonal in that basis. In particular, all its eigenvalues are on the unit circle. On the other hand, the eigenvalues of this matrix are minus roots of the polynomial $\hat{m}(z)$ [29, Lemmas 1,2]. Thus, all roots of $\hat{m}(z)$ and hence all roots of m are on the unit circle.

□

9.5. The Euler binary partition function

For arbitrary $r \in \mathbb{N} \cup \{\infty\}$, the Euler binary partition function $b(k) = b_r(k)$ is defined on the set of nonnegative integers k as the total number of different binary expansions $k = \sum_{j=0}^{\infty} d_j 2^j$, where the “digits” d_j take values from the set $D = \{d \in \mathbb{Z} \mid 0 \leq d < r\}$. For instance, if $r = 2$, then $D = \{0, 1\}$, and every number k has a unique binary expansion, therefore $b_2(k) \equiv 1$. If $r \geq 3$, then a binary expansion may not be unique, and a natural question arises how does $b_r(k)$ grow with k ? The asymptotic behavior of $b_r(k)$ as $k \rightarrow \infty$ was studied in various contexts by L. Euler, K. Mahler, N.G. de Bruijn, D.E. Knuth, and others (see [31] for bibliography and historical comments). Leonard Euler first considered the function $b_{\infty}(k)$ in connection with the power series $\sum_{k=0}^{\infty} b_{\infty}(k) z^k = \prod_{j=0}^{\infty} \frac{1}{1-z^{2^j}}$. In 1940 K. Mahler proved that $b_{\infty}(k) \asymp k^{\frac{1}{2} \log_2 k}$, this asymptotic relation was sharpened by N.G. de Bruijn (1948) and by D.E. Knuth (1966). The case of finite r was first systematically analyzed by B.Reznick in 1990 [31]. He showed that for every even r , we have $b_r(k) \asymp k^{\log_2(r/2)}$, i.e., the partition function grows polynomially in k . For odd r , however, the asymptotics of $b_r(k)$ as $k \rightarrow \infty$ is not that regular. Denote

$$p_1(r) = \liminf_{k \rightarrow \infty} \log b_r(k) / \log k; \quad p_2(r) = \limsup_{k \rightarrow \infty} \log b_r(k) / \log k. \quad (10)$$

Thus, for even r we have $p_1 = p_2 = \log_2(r/2)$. However, already for $r = 3$, the lower and upper exponents of growth do not coincide $p_1(3) = 0$ and $p_2(3) = \log_2 \frac{\sqrt{5}+1}{2}$. B.Reznick in [31] formulated the following problem: is it true that $p_1 < p_2$ for all even r ? Applying Theorem 5 we obtain an affirmative answer:

Theorem 13 *For every odd r we have $p_1(r) < p_2(r)$.*

Thus, for every odd r the lower and upper exponents of growth of the function $b_r(k)$ do not coincide. To prove the theorem we need several auxiliary results and observations. In [27] it was shown that $p_1 = \log_2 \check{\rho}(D_0, D_1)$ and $p_2 = \log_2 \rho(D_0, D_1)$, where D_0, D_1 are $(r-1) \times (r-1)$ -matrices defined as follows:

$$(D_s)_{ij} = \begin{cases} 1, & \text{if } 1-s \leq 2i-j \leq r-s, \\ 0, & \text{otherwise} \end{cases} \quad (11)$$

(here $s = 0, 1$ and $i, j \in \{1, \dots, r-1\}$). For example, for $r = 5$ we have the following 4×4 -matrices:

$$D_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider the matrices $B_s = \frac{2}{r} D_s$, $s = 0, 1$. Note that the matrix $\bar{B} = \frac{1}{2}(B_0 + B_1)$ is column-stochastic, and hence $\rho_1(B_0, B_1) = \rho(\bar{B}) = 1$. Therefore, $\check{\rho}(B_0, B_1) \leq 1 \leq \rho(B_0, B_1)$. We see that $p_1 = p_2$ if and only if $\check{\rho} = \rho = 1$. Thus, we have proved

Proposition 15 *We have $p_1(r) = p_2(r)$ if and only if the family $\{B_0, B_1\}$ is c.s.r.*

From this fact it is already clear why $p_1(r) = p_2(r)$ for even r . Indeed, each column of the matrices D_0, D_1 , has exactly $\frac{r}{2}$ ones, hence B_0, B_1 are both column-stochastic, and the family $\{B_0, B_1\}$ is c.s.r. Theorem 13 states that that this family is not c.s.r. for odd r .

Proof of Theorem 13. Comparing formulas (11) and (9) we see that our matrices B_0, B_1 correspond to the refinement equation $T\varphi = \varphi$ with the transition operator $[T\varphi](t) = \frac{2}{r} \sum_{k=0}^{r-1} \varphi(2t - k)$. The associated generating polynomial is $m(z) = \frac{1}{r} \sum_{k=0}^{r-1} z^k = \frac{z^r - 1}{r(z - 1)}$. If r is odd, then $m(-1) = \frac{1}{r} \neq 0$. Moreover, $(z^r - 1) + ((-z)^r - 1) \equiv -2$, hence the values $m(z)$ and $m(-z)$ cannot vanish simultaneously. Thus, by Theorem A, the refinement equation does not have a nontrivial summable solution.

Assume now that $\{B_0, B_1\}$ is c.s.r. The matrix \bar{B} has strictly positive elements on the main diagonal and on both the neighboring diagonals, hence \bar{B} is positively irreducible, and so is the pair $\{B_0, B_1\}$. Theorem 5 implies that B_0 and B_1 possess a common invariant affine subspace $V \subset \mathbb{R}^{r-1}$ such that the set $\mathbf{P} = V \cap \mathbb{R}^{r-1}$ is bounded. Since $B_i \mathbf{P} \subset \mathbf{P}$, it follows that the transition operator respects the set of functions $f \in L_1(\mathbb{R})$ such that $v(t) = (\varphi(t), \dots, \varphi(t - r + 2))^T \in \mathbf{P}$ for almost all $t \in [0, 1]$. Taking an arbitrary initial function f_0 from this set, we see that $T^n f_0$ belongs to this set for each n , and hence, so does the limit $\varphi = \lim_{n \rightarrow \infty} T^n f_0$, which is, by Theorem A, a solution of the refinement equation. For this solution $v(t) \in \mathbf{P}$ a.e., therefore $\varphi \in L_\infty[0, 1]$. This contradiction completes the proof. □

10. Appendix

Proof of Proposition 2. The necessity is obvious, let us prove sufficiency. If the spectral radii of matrices from \mathcal{S} are bounded, then they are bounded by one. Assume to the contrary that there are $A_i \in \mathcal{S}$ such that $\|A_i\| \rightarrow \infty$ as $i \rightarrow \infty$. Since \mathcal{S} has a finite irreducible subset, we assume that the sequence $\{A_i\}_{i=1}^\infty$ starts with this subset, i.e., the set $\{A_1, \dots, A_n\}$ is irreducible. For each $k \geq \mathbb{N}$ we denote by \mathcal{S}_k the semigroup generated by the matrices $\{A_1, \dots, A_k\}$. This set is irreducible, and since the spectral radii of all products of these matrices do not exceed one, their joint spectral radius is at most one. Hence, by Barabanov's theorem [1], there is a norm $\|\cdot\|_k$ in \mathbb{R}^d such that $\|A_i\|_k \leq 1$, $i = 1, \dots, k$. We normalize this norm by the condition that its maximum on the unit Euclidean ball is one. By the compactness, there is a limit point $f(\cdot)$ of this sequence of norms. Passing to a subsequence, it may be assumed that $\|x\|_k \rightarrow f(x)$, $x \in \mathbb{R}^d$, as $k \rightarrow \infty$. Obviously, f is a nonnegative convex symmetric homogeneous functional. The set of points $x \in \mathbb{R}^d$ such that $f(x) = 0$ is a proper linear subspace of \mathbb{R}^d invariant with respect to all A_i , which contradicts the irreducibility. Hence, $\|x\| > 0$ for $x \neq 0$, and therefore f is a norm. The induced operator norm of all A_i is at most one. Hence, by the equivalence of norms, $\|A_i\|$ cannot tend to infinity as $i \rightarrow \infty$. □

Proof of Proposition 5. If \mathcal{S} is positively irreducible, then it contains a finite positively irreducible subset. Therefore, without loss of generality it can be assumed that \mathcal{S} is generated by a finite matrix family $\mathcal{A} = \{A_1, \dots, A_m\}$. The matrix $\bar{A} = \frac{1}{m} \sum_{i=1}^m A_m$ is positively irreducible, and $\bar{A}V \subset V$, hence V intersects the interior of \mathbb{R}_+^d . Otherwise, the set \mathbf{P} is contained in some coordinate subspace which is invariant for \bar{A} . If \mathbf{P} is unbounded, then it contains a ray. A parallel ray starting at the origin is contained in the subspace \tilde{V} (the linear part of V). Consequently, the cone $K = \mathbb{R}_+^d \cap \tilde{V}$ is nontrivial. Since $\bar{A}K \subset K$, it follows from the Krein-Rutman theorem [15] that \bar{A} possesses an eigenvector $v \in K$. Hence, v is a Perron-Frobenius eigenvector of the matrix \bar{A} , because a positively irreducible matrix has a unique, up to normalization, nonnegative eigenvector. Thus, V is parallel to the vector v . Let \bar{A}^T be the transpose matrix and v^* be its Perron-Frobenius eigenvector. For sufficiently large $s > 0$, the set $V_s = \{x \in \mathbb{R}_+^d \mid (v^*, x) = s\}$ intersects V by some nonempty set \mathbf{G} . Clearly, $\bar{A}\mathbf{G} \subset \mathbf{G}$. Since \bar{A}^T is positively irreducible, it follows that $v^* > 0$, and hence the set \mathbf{G} is bounded. The operator \bar{A} respects the cone $K_s = \{tx \mid t \geq 0, x \in \mathbf{G}\}$, and consequently, $v \in K_s$. Hence, $\tau v \in V$ for some $\tau > 0$. However, $\tau v \in \tilde{V}$, and so $0 = (\tau v - \tau v) \in V$, which contradicts the assumption. \square

Proof of Theorem 5. Sufficiency. Suppose \mathcal{S} has a common invariant affine subspace V satisfying all the assumptions. By Proposition 5, the set $\mathbf{P} = V \cap \mathbb{R}_+^d$ is bounded and contains some interior point $z \in \text{int}(\mathbb{R}_+^d)$. There are positive constants C_1, C_2 such that $C_1 \|Bz\| \leq \|B\| \leq C_2 \|Bz\|$ for each nonnegative matrix B . Since the set $\{Az \mid A \in \mathcal{S}\}$ is contained in \mathbf{P} , we see that $C_1 \inf_{x \in \mathbf{P}} \|x\| \leq \|A\| \leq C_2 \sup_{x \in \mathbf{P}} \|x\|$ for all $A \in \mathcal{S}$. Thus, \mathcal{S} is bounded and separated from zero, hence it is c.r.s.

Necessity. Since \mathcal{S} is irreducible, it contains a finite positively irreducible subset \mathcal{B} . Every finite subset $\mathcal{A} \subset \mathcal{S}$ that contains \mathcal{B} is also irreducible. If every such a subset \mathcal{A} possesses a common invariant affine subspace satisfying all the assumptions, then so does the set \mathcal{S} . Thus, it suffices to prove the necessity for finitely generated c.s.r. semigroups. Assume $\mathcal{S} = \mathcal{S}_{\mathcal{A}}$ for some $\mathcal{A} = \{A_1, \dots, A_m\}$. For a nonnegative matrix family \mathcal{A} we have $\rho_1(\mathcal{A}) = \rho(\bar{A})$, where $\bar{A} = \frac{1}{m} \sum_{i=1}^m A_m$ (see [30]). Hence, in our case, $\rho(\bar{A}) = 1$. Furthermore, since \mathcal{A} is positively irreducible, it follows that there is a norm $\|\cdot\|$ such that $\max_{i=1, \dots, m} \|A_i x\| = \rho(\mathcal{A}) \|x\|$, for all $x \in \mathbb{R}_+^d$ (see [10, Theorem 3]). In our case $\rho(\mathcal{A}) = 1$, hence

$$\max_{d_1, \dots, d_k} \|A_{d_k} \cdots A_{d_1} x\| = \|x\|, \quad x \in \mathbb{R}_+^d, k \in \mathbb{N}. \quad (12)$$

Let $v \in \text{int}(\mathbb{R}_+^d)$, $\|v\| = 1$, be a Perron-Frobenius eigenvector of \bar{A} . We have $v = \bar{A}v$, therefore

$$v = \bar{A}^k v = m^{-k} \sum_{d_1, \dots, d_k} A_{d_k} \cdots A_{d_1} v, \quad k \in \mathbb{N}. \quad (13)$$

Let $L_v = \{x \in \mathbb{R}^d \mid (p, x) = 1\}$ be the hyperplane of support for the unit ball of the norm $\|\cdot\|$ passing through the point v . Thus, $(p, v) = 1$ and $(p, x) \leq 1$ for every $x \in \mathbb{R}^d$, $\|x\| \leq 1$.

Substituting in (13), we get

$$1 = (p, v) = m^{-k} \sum_{d_1, \dots, d_k} (p, A_{d_k} \cdots A_{d_1} v). \quad (14)$$

By (12) we have $\|A_{d_k} \cdots A_{d_1} v\| \leq \|v\| = 1$, therefore, $(p, A_{d_k} \cdots A_{d_1} v) \leq 1$, for every d_k, \dots, d_1 . Combining with (14) we obtain $(p, A_{d_k} \cdots A_{d_1} v) = 1$, for every d_k, \dots, d_1 , which means that all the points $A_{d_k} \cdots A_{d_1} v$ belong to the affine hyperplane L_v . Let V be the affine hull of the points v and $A_{d_k} \cdots A_{d_1} v$ over all possible products of lengths $k \in \mathbb{N}$. Clearly, $A_i V \subset V$, $i = 1, \dots, m$ and $v \in V \subset L_v$. Thus, V is a required proper affine subspace. □

Proof of Lemma 1. Let $C \in \mathcal{S}$ be an operator of the minimal rank r and let L be the range of C . Clearly, C is nonsingular on L , otherwise the rank of C^2 is smaller than r . Hence $\dim L = r$, the kernel K of C is of dimension $d - r$ and does not intersect L . Since C is a direct sum of a nilpotent operator and of an operator similar to orthogonal [24, Theorem 2.5], the operator $C|_L$ is similar to orthogonal. Hence, powers of this operator come arbitrarily close to the identity operator. Therefore, some powers C^m are arbitrarily close to the projector P onto L parallel to the subspace K . Since \mathcal{S} is closed, it contains P . Consider the semigroup $\mathcal{S}_P = \{(PA)|_L \mid A \in \mathcal{S}\}$. The range of the operator $(PA)|_L$ coincides with the range of PAP , whose dimension is r , otherwise $\text{rank}(PAP) < r$. Whence, $(PA)|_L$ is nonsingular. Moreover, $\rho((PA)|_L) = \rho(PAP) = 1$. Thus, the semigroup \mathcal{S}_P consists of nonsingular matrices and has constant spectral radius. Let us show that \mathcal{S}_P is irreducible. If, to the contrary, there is a nontrivial subspace $L' \subset L$ such that $PA(L') \subset L'$ for all $A \in \mathcal{S}$, then $AL' \subset (L' + K)$. Hence, the linear span of the set $\{AL' \mid A \in \mathcal{S}\}$, which is a common invariant subspace for all elements of \mathcal{S} , is contained in the subspace $L' + K$. This contradicts the irreducibility of \mathcal{S} . Thus, \mathcal{S}_P is irreducible, is of constant spectral radius, and consists of nonsingular operators. By Theorem 2, one can choose a basis in L so that all operators of \mathcal{S}_P become orthogonal. Now we complement this basis to a basis of \mathbb{R}^d so that K becomes orthogonal to L . In the new basis, P is an orthogonal projector, which concludes the proof. □

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